KIX1002: ENGINEERING MATHEMATICS 2

TUTORIAL 12: Partial Differential Equation I

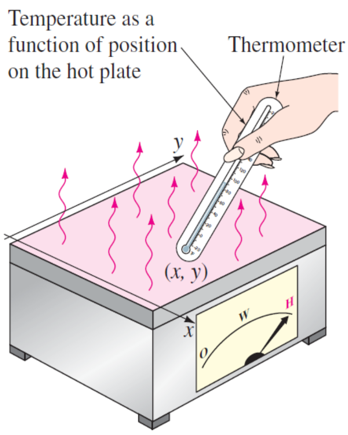
1) Categorize the following equations in terms of its order, linearity, and homogeneity.
   (a) $\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + 1 = 0$
   (b) $\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} + xu = 0$
   (c) $(u_{tt})^3 - u_{xx} + x^2 = 0$
   (d) $u_x - u_{xxy} + uu_y = 0$
   (e) $u' + u'''' + \sqrt{1+u} = 0$

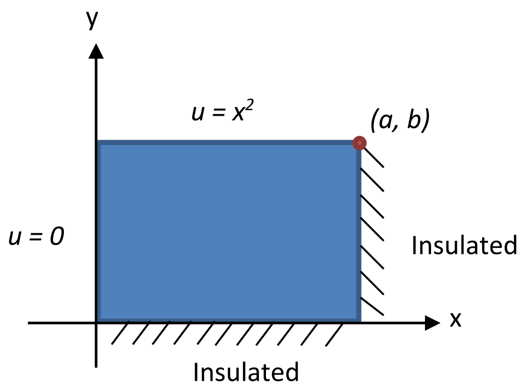
Classify the category of the following PDEs and solve the PDEs using separation of variable method.

2) $u_{xx} + u_{yy} = 0$
3) $u_t - u_{xx} = 0$
4) $u_t - \frac{1}{16} u_{xx} = 0$
5) $u_{tt} - u_{xx} = 0$
6) $u_{xx} = 3u_{yy}$
7) $u_{tt} = 4a^2 u_{xx}$

8) Set up the boundary and initial conditions from the given statement/figure that describe the scenario. Consider a hot place of area *(xy)*, set up the boundary value problem for the steady-state temperature over the *x* and *y* location, i.e. *u(x,y)*.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



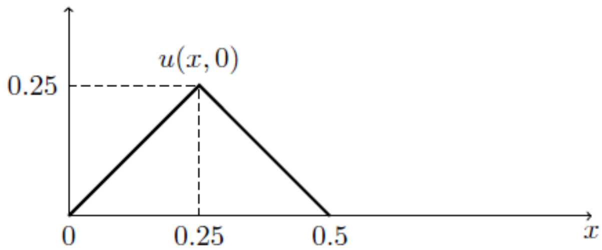


9) Set up the boundary and initial conditions from the given statement/figure that describe the scenario. A metal rod coincides with the interval [*0, L*] on the x-axis with both ends fixed at 0℃. It has an initial temperature of $\cos(\frac{\pi}{L}x)$. Set up the boundary value problem for the temperature *u(x, t).*

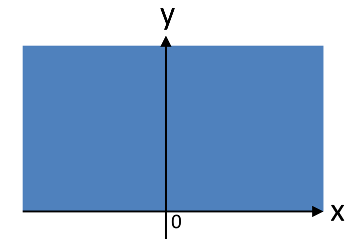
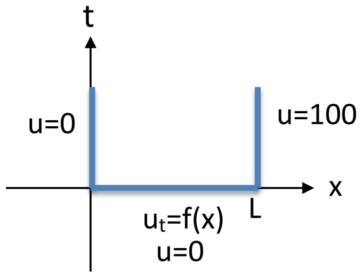
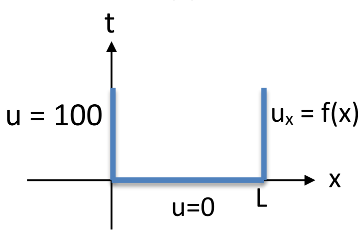
$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

10) Set up the boundary value problem for the displacement *u(x, t)* when a string with length, L = 1, is fixed at the two ends on the x-axis with the initial shape shown as the graph below. The string is released from rest.

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$



11) Match the given situations to their corresponding equations and conditions.

Situation		Equation		Condition
(a)  $u_y(x, 0) - u(x, 0) = f(x)$		***(d)*** *1D Heat Equation* $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$		***(g)*** $-\infty < x < \infty$ $y > 0$ $\frac{\partial u(x,0)}{\partial y} - u(x, 0) = f(x)$
(b) 		***(e)*** *2D Laplace Equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$		***(h)*** $u(0, t) = 100, t > 0$ $u(x, 0) = 0, 0 < x < L$ $\left.\frac{\partial u}{\partial x}\right\|_{x=L} = f(x), t > 0$
(c) 		***(f)*** *1D Wave Equation* $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$		***(i)*** $\left.\begin{matrix} u(0, t) = 0 \\ u(L, t) = 100 \end{matrix}\right\} t > 0$ $\left.\begin{matrix} u(x, 0) = 0 \\ \left.\frac{\partial u}{\partial t}\right\|_{t=0} = f(x) \end{matrix}\right\} 0 < x < L$

Short Answer for Self-Check:

Q1 (a) Second order, linear, and non-homogeneous PDE

Q1 (b) Second order, linear, and homogeneous PDE

Q1 (c) Second order, non-linear, and non-homogeneous PDE

Q1 (d) Third order, non-linear, and homogeneous PDE

Q1 (e) Fourth order, non-linear, and non-homogeneous PDE

Q2 Elliptic PDE

$$
\begin{aligned}
u(x,y) = &\left(c_1 + c_2 y\right)\left(c_3 + c_4 x\right) \\
&+\left(c_5 \cos(\alpha y) + c_6 \sin(\alpha y)\right)\left(c_7 \cosh(\alpha x) + c_8 \sinh(\alpha x)\right) \\
&+\left(c_9 \cosh(\alpha y) + c_{10} \sinh(\alpha y)\right)\left(c_{11} \cos(\alpha x) + c_{12} \sin(\alpha x)\right)
\end{aligned}
$$

Q3 Parabolic PDE

$$
u(x,t) =
$$

$$
A_1 x + B_1 + e^{\alpha^2 t}\left(A_2 cosh(\alpha x) + B_2 sinh(\alpha x)\right) + e^{-\alpha^2 t}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)
$$

Q4 Parabolic PDE

$$
u(x,t) =
$$

$$
A_1 x + B_1 + e^{\alpha^2 t}\left(A_2 cosh(4\alpha x) + B_2 sinh(4\alpha x)\right) + e^{-\alpha^2 t}\left(A_3 cos(4\alpha x) + B_3 sin(4\alpha x)\right)
$$

Q5 Hyperbolic PDE

$$
u(x,t) =
$$

$$
\begin{aligned}
&\left(c_1 + c_2 t\right)\left(c_3 + c_4 x\right) + \left(c_5 cos\,h(\alpha t) + c_6 sin\,h(\alpha t)\right)\left(c_7 cos\,h(\alpha x) + c_8 sin\,h(\alpha x)\right) \\
&+ \left(c_9 cos(\alpha t) + c_{10} sin(\alpha t)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\right)
\end{aligned}
$$

Q6 Elliptic PDE

$$
\begin{aligned}
&u(x,y) \\
&= \left(c_1 + c_2 y\right)\left(c_3 + c_4 x\right) + \left(c_5 \cos(\alpha y) + c_6 \sin(\alpha y)\right)\left(c_7 \cosh(\sqrt{3}\alpha x) + c_8 \sinh(\sqrt{3}\alpha x)\right) \\
&+ \left(c_9 \cosh(\alpha y) + c_{10} \sinh(\alpha y)\right)\left(c_{11} \cos(\sqrt{3}\alpha x) + c_{12} \sin(\sqrt{3}\alpha x)\right)
\end{aligned}
$$

Q7 Hyperbolic PDE

$$u(x,t) =$$
$$\left(c_1 + c_2 t\right)\left(c_3 + c_4 x\right) + (c_5 cos\,h(\alpha t) + c_6 sin\,h(\alpha t))(c_7 cos\,h(2a\alpha x) + c_8 sin\,h(2a\alpha x)$$
$$+ \left(c_9 cos(\alpha t) + c_{10} sin(\alpha t)\right)\left(c_{11} cos(2a\alpha x) + c_{12} sin(2a\alpha x)\right)$$

Q8:

$u(0,\ y) = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$ for $0 < y < b$

$u(x,\ b) = x^2,\ \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$ for $0 < x < a$

Q9:

$u(0,\ t) = 0$, $u(L,$ t$)= 0$ for $t > 0$

$$u(x,\ 0) = \cos(\frac{\pi}{L}x) \qquad \text{for } 0 < x < L$$

Q10:

$$u(0,\ 0) = 0,\ u(L,\ 0) = 0$$

$$u(x,\ 0) = f(x) = \begin{cases} x & \text{for } 0 < x < 1/4 \\ \frac{1}{2} - x & \text{for } 1/4 < x < 1/2 \\ 0 & \text{for } 1/2 < x < 1 \end{cases}$$

$$u_t(x,\ 0) = 0$$

Q11:

(a) – (e) – (g)
(b) – (f) – (i)
(c) – (d) – (h)