KIX1002: ENGINEERING MATHEMATICS 2

TUTORIAL 13: Partial Differential Equation II

The general solutions of the following PDEs have been found in Tutorial 12, Question 2 –5. Continue to find the particular solution for each case based on the given boundary and initial conditions.

Laplace equation, $u_{xx} + u_{yy} = 0$

General solution: $u(x, y)$

$$= (c_1 + c_2 y)(c_3 + c_4 x) + (c_5 \cos(\alpha y) + c_6 \sin(\alpha y))(c_7 \cosh(\alpha x) + c_8 \sinh(\alpha x)) + (c_9 \cosh(\alpha y) + c_{10} \sinh(\alpha y))(c_{11} \cos(\alpha x) + c_{12} \sin(\alpha x))$$

Subject to the given boundary conditions:

1) $u(0, y) = 0,\quad u(a, y) = 0,\quad 0 < y < b$
   $u(x, 0) = 0,\quad u(x, b) = f(x),\quad 0 < x < a$

2) $u(0, y) = 0,\quad u(1, y) = 1 - y,\quad 0 < y < 1$
   $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0,\quad 0 < x < 1$

3) $\left.\frac{\partial u}{\partial x}\right|_{y=x} = u(0, y),\quad u(\pi, y) = 1,\quad 0 < y < \pi$
   $u(x, 0) = 0,\quad u(x, \pi) = 0,\quad 0 < x < \pi$

4) $u(0, y) = 0,\quad u(a, y) = 50,\quad 0 < y < b$
   $u(x, 0) = 0,\quad u(x, b) = 0,\quad 0 < x < a$

5) $u(0, y) = 0,\quad u(a, y) = 50,\quad 0 < y < b$
   $u(x, 0) = 0,\quad u(x, b) = f(x),\quad 0 < x < a$

6) Heat equation, $u_t - u_{xx} = 0$

   General solution: $u(x, t)$

$$= A_1 x + B_1 + e^{\alpha^2 t}(A_2 cosh(\alpha x) + B_2 sinh(\alpha x)) + e^{-\alpha^2 t}(A_3 cos(\alpha x) + B_3 sin(\alpha x))$$

   Subject to the given boundary conditions:

   $u(0, t) = 0,\quad u(2, t) = 0,\quad t > 0$
   $u(x, 0) = \sin\frac{\pi}{2}x,\quad 0 < x < 2$

7) Heat equation, $u_t - \frac{1}{16}u_{xx} = 0$

General solution: $u(x,t)$

$$= A_1x + B_1 + e^{\alpha^2 t}\left(A_2 cosh(4\alpha x) + B_2 sinh(4\alpha x)\right)$$
$$+ e^{-\alpha^2 t}\left(A_3 cos(4\alpha x) + B_3 sin(4\alpha x)\right)$$

Subject to the given boundary conditions:

$u(0,\ t) = 0,$ $\quad u(1,\ t) = 0,$ $\quad t > 0$

$u(x,\ 0) = 2\sin 2\pi x,$ $\quad 0 < x < 1$

8) Wave equation, $u_{tt} - u_{xx} = 0$

General solution: $u(x,t)$

$$= (c_1 + c_2 t)(c_3 + c_4 x)$$
$$+ (c_5 cos\,h(\alpha t) + c_6 sin\,h(\alpha t))(c_7 cos\,h(\alpha x) + c_8 sin\,h(\alpha x))$$
$$+ (c_9 cos(\alpha t) + c_{10} sin(\alpha t))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$$

Subject to the given boundary conditions:

$u(0,\ t) = 0,$ $\quad u(1,\ t) = 0,$ $\quad t > 0$

$u(x,\ 0) = \sin \pi x,$ $\quad \frac{\partial u}{\partial t}(x,0) = 0$ $\quad 0 < x < 1$

9) Continue to estimate the solution of Q2 at $(0.9,0.1)$ by using 2 summation terms. Assume the actual answer is 0.74045 and the allowable percentage of error is 1%. Please comment if the estimation is acceptable or not. If no, please suggest how to improve the solution.

10) Identify the eigenvalue and eigenvector of PDE solutions that has been solved in Q2.

Short Answer for Self-Check:

Q1: $u_{total}(x,y) = \sum_{n=1}^{\infty}\left(\frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\, sin\, n\frac{\pi}{a}x\, dx\, \sinh(\frac{n\pi}{a}y)\right)\ \left(\sin(\frac{n\pi}{a}x)\right)$

Q2: $u_{total}(x,y) = \frac{1}{2}x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(1-(-1)^n)}{n^2\sinh(n\pi)}\sinh(n\pi x)\right)\ (\cos(n\pi y))$

Q3: $u_{total}(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(1-(-1)^n)}{n}\frac{n\cosh(nx)+\sinh(nx)}{n\cosh(n\pi)+\sinh(n\pi)}\right)(\sin(ny))$

Q4: $u_{total}(x,y) = \sum_{n=1}^{\infty}\left(\frac{100}{n\pi\sinh(\frac{n\pi}{b}a)}[1-(-1)^n]\sinh(\frac{n\pi}{b}x)\right)\ \left(\sin(\frac{n\pi}{b}y)\right)$

Q5: $u_{total}(x,y) = \sum_{n=1}^{\infty}\left(\frac{100}{n\pi\sinh\left(\frac{n\pi}{b}a\right)}[1-(-1)^n]\sinh\left(\frac{n\pi}{b}x\right)\right)\ \left(\sin\left(\frac{n\pi}{b}y\right)\right) +$
$\sum_{n=1}^{\infty}\left(\frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\, sin\, n\frac{\pi}{a}x\, dx\, \sinh(\frac{n\pi}{a}y)\right)\ \left(\sin(\frac{n\pi}{a}x)\right)$

Q6: $u_{total}(x,t) = e^{-\frac{\pi^2}{4}t}\ \left(\sin(\frac{\pi}{2}x)\ \right)$

Q7: $u_{total}(x,t) = 2e^{-\frac{\pi^2}{4}t}\ (\sin(2\pi x))$

Q8: $u_{total}(x,t) = (cos(\pi t))\ (sin(\pi x))$

Q9: $u_{total}(0.9,0.1)$ ≈0.73107

Q10: For case 1, eigenvalue, *λ=0 ; eigenfunction of PDE:* $u_1 = (A_1 + B_1x)$

For case 2, eigenvalue, *λ=*-$(n\pi)^2$ *; eigenfunction of PDE:*

$u_{2,\mathrm{n}}(x,y) = (\cos(n\pi y))\left(A_{2,n}cosh(n\pi x) + B_{2,n}sinh(n\pi x)\right)$ *where n=1, 2, 3, …*

No eigenvalue & eigenfunction for case 3.