APPENDIX 12.1 SOLVE THE PDE LIKE ODE – EXTRA INFO

We can solve the PDE like the ODE when there is only one-independent-variable derivative in the equation. For example:

$$\frac{\partial^2}{\partial t^2}\{u(x,t)\} - u(x,t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} - u(x,t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} + \frac{\partial}{\partial x}\{u(x,t)\} - u(x,t) = 0$$

There are similarity and differences between the ODE and PDE. For example:

Case #1: 2 Distinct Real Roots (Let dependent variable $= u$; independent variables $= x, t$)

Solution for linear homogeneous ODE	**Solution for linear homogeneous PDE**
Solve $\frac{d^2}{dt^2}\{u(t)\} - u(t) = 0$ $Let\ u(t) = e^{rt}$ $r^2e^{rt} - e^{rt} = 0$ $(r^2 - 1)e^{rt} = 0$ The solution $e^{rt} \neq 0$ Hence, Characteristic equation: $(r^2 - 1) = 0$ $r^2 = 1$ $r = \pm 1$ We have 2 independent solutions, i.e. e^t, e^{-t} Using linear superposition: $\therefore u(t) = c_1e^{-t} + c_2e^t$ Boundary conditions: $u(0) = 1, u(1) = 0$ $\therefore u(t) = 1.157e^{-t} - 0.157e^t$	Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} - u(x,t) = 0$ $Let\ u(x,t) = e^{rt}$ $r^2e^{rt} - e^{rt} = 0$ $(r^2 - 1)e^{rt} = 0$ The solution $e^{rt} \neq 0$ Hence, Characteristic equation: $(r^2 - 1) = 0$ $r^2 = 1$ $r = \pm 1$ We have 2 independent solutions, i.e. e^t, e^{-t} Using linear superposition: $\therefore u(x,t) = c_1(x)e^{-t} + c_2(x)e^t$ Boundary conditions: $u(x,0) = x, u(x,1) = 0$ $\therefore u(x,t) = (1.157x)e^{-t} - (0.157x)e^t$

Note: ODE has arbitrary constant (e.g. c_1) while PDE has arbitrary function (e.g. c_1(x))

Case #2: 2 Distinct Complex Roots (Let dependent variable = u ; independent variables = x, t)

Solution for linear homogeneous ODE	Solution for linear homogeneous PDE
Solve $\frac{d^2}{dt^2}\{u(t)\} + u(t) = 0$ $Let\ u(t) = e^{rt}$ Hence, Characteristic equation: $(r^2 + 1) = 0$ $r^2 = -1$ $r = \pm\sqrt{1} = \pm i$ We have 2 independent solutions, i.e. e^{it}, e^{-it} Using linear superposition: $\therefore u(t) = c_1 e^{-it} + c_2 e^{it}$ $= A_1 cost + A_2 sint$	Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} + u(x,t) = 0$ $Let\ u(x,t) = e^{rt}$ Hence, Characteristic equation: $(r^2 + 1) = 0$ $r^2 = -1$ $r = \pm\sqrt{1} = \pm i$ We have 2 independent solutions, i.e. e^{it}, e^{-it} Using linear superposition: $\therefore u(x,t) = c_1(x) e^{-it} + c_2(x) e^{it}$ $= A_1(x) cost + A_2(x) sint$

Note: ODE has arbitrary constant (e.g. c_1) while PDE has arbitrary function (e.g. $c_1(x)$)

Case #3: 2 Identical Roots (Let dependent variable = u ; independent variables = x, t)

Solution for linear homogeneous ODE	Solution for linear homogeneous PDE
Solve $\frac{d^2}{dt^2}\{u(t)\} + 2\frac{d}{dt}\{u(t)\} + u(t) = 0$ $Let\ u(t) = e^{rt}$ Characteristic equation: $(r^2 + 2r + 1) = 0$ $(r+1)(r+1) = 0$ $r = -1$ We have 2 dependent solutions, i.e. e^{-t}, e^{-t} Treatment: Multiply its independent variable New solutions: e^{-t}, te^{-t} Using linear superposition: $\therefore u(t) = c_1 e^{-t} + c_2 te^{-t}$	Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} + 2\frac{\partial}{\partial t}\{u(x,t)\} + u(x,t) = 0$ $Let\ u(x,t) = e^{rt}$ Characteristic equation: $(r^2 + 2r + 1) = 0$ $(r+1)(r+1) = 0$ $r = -1$ We have 2 dependent solutions, i.e. e^{-t}, e^{-t} Treatment: Multiply its independent variable New solutions: e^{-t}, te^{-t} Using linear superposition: $\therefore u(x,t) = c_1(x) e^{-t} + c_2(x) te^{-t}$

Note: ODE has arbitrary constant (e.g. c_1) while PDE has arbitrary function (e.g. $c_1(x)$)

More examples:

Solve $u_{xx} - u = 0$, where $u = u(x, y)$

Solution:

Since $u = u(x, y)$
Dependent variable: u
Independent variable: x, y

One-independent-variable derivative, i.e. x –derivative, where x as the variable while y as the constant, thus we can solve the PDE like ODE.

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x, y)\}$
$u_{xx} - u = \frac{\partial^2}{\partial x^2}\{u(x, y)\} - u(x, y) = 0$

Similar to ODE, $u''(x) - u(x) = 0$ where $u(x) = e^{rx}$
Characteristic equation, $r^2 - 1 = 0$
2 real roots: $r_1 = 1$, $r_2 = -1$
Solution of PDE: $u(x, y) = c_1(y)e^{x} + c_2(y)e^{-x}$, where $c_1(y), c_2(y)$= arbitrary functions

Solve $u_{yy} - u = 0$, where $u = u(x, y)$

Solution:

Since $u = u(x, y)$
Dependent variable: u
Independent variable: x, y

One-independent-variable derivative, i.e. y –derivative, where y as the variable while x as the constant, thus we can solve the PDE like ODE.

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x, y)\}$
$u_{xx} - u = \frac{\partial^2}{\partial x^2}\{u(x, y)\} - u(x, y) = 0$

Similar to ODE, $u''(y) - u(y) = 0, where\, u(y) = e^{ry}$
Characteristic equation, $r^2 - 1 = 0$
2 real roots: $r_1 = 1$, $r_2 = -1$
Solution of PDE: $u(x, y) = c_1(x)e^{x} + c_2(x)e^{-x}$, where $c_1(x), c_2(x)$= arbitrary functions

Note that this approach can't solve the PDE problems if there are two-independent-variable derivative.

For example:

$$\frac{\partial^2}{\partial x \partial y}\{u(x, t)\} + \frac{\partial}{\partial x}\{u(x, t)\} - u(x, t) = 0$$

$$\frac{\partial^2}{\partial x^2}\{u(x, t)\} + \frac{\partial}{\partial y}\{u(x, t)\} - u(x, t) = 0$$

APPENDIX 12.2 SOLVE THE PDE BY DIRECT INTEGRATION– EXTRA INFO

We can solve the PDE by direct integration when there is only one derivative component in the equation. For example:

$$\frac{\partial^2}{\partial t^2}\{u(x,t)\} = 5xe^{-10t}$$

$$\frac{\partial}{\partial t}\{u(x,t)\} = 5xe^{-10t}$$

$$\frac{\partial^2}{\partial t \partial x}\{u(x,t)\} = 5xe^{-10t}$$

- Using Direct integration on ODE vs PDE

Integration in ODE (Arbitrary Constants)	**Integration in PDE (Arbitrary Functions)**
Solve $\frac{d^2}{dt^2}\{u(t)\} = 0$ Integrate both sides, $\int \frac{d^2}{dt^2}\{u(t)\}dt = \int 0dt$ $\frac{d}{dt}\{u(t)\} = 0t + c_1$ Integrate both sides again, $\int \frac{d}{dt}\{u(t)\}dt = \int c_1 dt$ $\therefore u(t) = c_1 t + c_2$ Where c_1 and c_2 are 2 arbitrary constants. These constants can be solved if 2 initial conditions or boundary conditions are provided. $\boldsymbol{Note}$: n^{th} order ODE will have n constants to be solved. (e.g. 2nd order ODE have 2 arbitrary constants)	Solve $\frac{\partial^2}{\partial t^2}\{u(x,t)\} = 0$ Integrate both sides, $\int \frac{\partial^2}{\partial t^2}\{u(x,t)\}dt = \int 0dt$ $\frac{\partial}{\partial t}\{u(x,t)\} = 0t + c_1(x)$ Integrate both sides again, $\int \frac{\partial}{\partial t}\{u(x,t)\}dt = \int c_1(x)dt$ $\therefore u(x,t) = c_1(x)t + c_2(x)$ Where $c_1(x)$and $c_2(x)$are 2 arbitrary functions of variable x. These functions can be solved if the initial conditions or boundary conditions are provided. $\boldsymbol{Note}$: n^{th} order PDE may need more than n arbitrary functions to be solved

- More examples:

Solve $\frac{\partial^2}{\partial x \partial y}\{u(x,y)\} = 0$

Solution for linear homogeneous PDE
Integrate both sides with respect to variable x, $\int \frac{\partial^2}{\partial x \partial y}\{u(x,y)\}dx = \int 0dx$ $\frac{\partial}{\partial y}\{u(x,y)\} = 0x + c_1(y)$

Integrate both sides with respect to variable y,

$\int \frac{\partial}{\partial y}\{u(x,y)\}dy = \int c_1(y)dy$

$\therefore u(x,y) = \int c_1(y)dy$

where $c_1(y)$ is the arbitrary function of variable y.

Solve $u_{xx} = 6xe^{-t}$ where $u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$; BC: $u(0,t) = t \; and \; u_x(0,t) = e^{-t}$

Solution:

- Dependent variable: u
- Independent variable: x, t

$$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\} = 6xe^{-t}$$

Note: One derivative component $\frac{\partial^2}{\partial x^2}$ and thus we can use direct integration

- Integrate the PDE with respect to variable x (Hence, variable t is constant)

$$\int \frac{\partial^2}{\partial x^2}\{u(x,t)\}dx = \int 6xe^{-t}dx$$

$$\frac{\partial}{\partial x}\{u(x,t)\} = \underbrace{6e^{-t}}_{\substack{treated\ as\ constant\\ when\ we\ integrated\\ wrt\ the\ variable\ x}} \int xdx = 6e^{-t}\frac{x^2}{2} + c_1(t)$$

- Integrate the PDE with respect to variable x (Hence, variable t is constant)

$$\int \frac{\partial}{\partial x}\{u(x,t)\}dx = \int 3e^{-t}x^2 + c_1(t)\,dx$$

General PDE solution: $u(x,t) = e^{-t}x^3 + xc_1(t) + c_2(t)$,

where the unknown arbitrary functions are $c_1(t)$ & $c_2(t)$.

Next, we continue to apply the boundary condition to solve the particular PDE solution.

$u(0,t) = t$

For x = 0: $u(x,t) = e^{-t}(0) + (0)c_1(t) + c_2(t) = t$

$\therefore c_2(t) = t$

$u_x(x,t) = \frac{\partial}{\partial x}[e^{-t}x^3 + xc_1(t) + c_2(t)] = 3e^{-t}x^2 + c_1(t)$

$u_x(0,t) = e^{-t}$

For x = 0: $u_x(x,t) = 3e^{-t}(0) + c_1(t) = e^{-t}$

$\therefore c_1(t) = e^{-t}$

Particular PDE solution: $u(x,t) = e^{-t}x^3 + xe^{-t} + t$

Solve $u_{xy} = sinxcosy$ where the boundary conditions are given:

When $y = \frac{\pi}{2}, u_x = 2x$

When $x = \pi, u = 2siny$

Solution:

- ***Dependent variable:*** u
- ***Independent variable:*** x & y

$u_{xy} = \frac{\partial^2}{\partial x \partial y}\{u(x,y)\} = sinxcosy$

Note: One derivative component $\frac{\partial^2}{\partial x \partial y}$ and thus we can use direct integration

- Integrate the PDE with respect to variable y (Hence, variable x is constant)

  $\int \frac{\partial^2}{\partial x \partial y}\{u(x,y)\}dy = \int sinxcosydy$

  $\frac{\partial}{\partial x}\{u(x,y)\} = sinx \int cosydy = sinxsiny + c_1(x)$

- Integrate the PDE with respect to variable x (Hence, variable y is constant)

  $\int \frac{\partial}{\partial x}\{u(x,y)\}dx = \int sinxsiny + c_1(x)dx$

  General PDE solution: $u(x,y) = -cosxsiny + \int c_1(x)dx + c_2(y)$

  where the unknown arbitrary functions are $c_1(x)$ & $c_2(y)$.

Next, we continue to apply the boundary condition to solve the particular PDE solution.

$u(\pi, y) = 2siny$

For $x = \pi$: $u(x,y) = -cos\pi siny + \int c_1(x)dx + c_2(y) = 2siny$

$\int c_1(x)dx + c_2(y) = siny$

$\therefore c_2(y) = siny - \int c_1(x)dx$ (Note: $c_2(y)$ has unknown $c_1(x)$ to be solved)

$u_x(x,y) = \frac{\partial}{\partial x}[-cosxsiny + \int c_1(x)dx + c_2(y)] = sinxsiny + c_1(x)$

$u_x\left(x, \frac{\pi}{2}\right) = 2x$

For $y = \frac{\pi}{2}$: $u_x(x,y) = sinxsin\frac{\pi}{2} + c_1(x) = 2x$

$\therefore c_1(x) = 2x - sinx$

Note: $c_1(x)$ is expressed in the variable x only

Substitute $c_1(x)$ into $c_2(y)$ equation where $u(\pi, y) = 2siny$

$$\begin{aligned} c_2(y) &= siny - \int 2x - sinxdx \\ &= siny - (x^2 + cosx) \\ &= siny - (\pi^2 + cos\pi) \\ &= siny + 1 - \pi^2 \end{aligned}$$

Note: $c_2(y)$ is expressed in the variable y only

Particular PDE solution: $u(x,y) = -cosxsiny + \int 2x - sinxdx + siny + 1 - \pi^2$

$= -cosxsiny + x^2 + cosx + siny + 1 - \pi^2$

APPENDIX 12.3 SOLVE THE PDE BY REDUCTION OF ORDER METHOD– EXTRA INFO

We can solve the PDE by reduction of order method when the order can be reduced by proper substitution.

For example:

$$\frac{\partial^2}{\partial x^2}\{u(x,t)\} + \frac{\partial}{\partial x}\{u(x,t)\} = 0$$

Order can be reduced by let $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$

$$\rightarrow \frac{\partial}{\partial x}\{p(x,t)\} + p(x,t) = 0$$

$$\frac{\partial^2}{\partial x \partial y}\{u(x,y)\} + \frac{\partial}{\partial x}\{u(x,y)\} = 0$$

Order can be reduced by let $g(x,y) = \frac{\partial}{\partial x}\{u(x,y)\}$

$$\rightarrow \frac{\partial}{\partial y}\{g(x,y)\} + g(x,y) = 0$$

Hence, we can solve the problem by using the integration, solve PDE like ode approach, etc.

For example, repeating the problem in Appendix 12.2:

Solve $u_{xx} = 6xe^{-t}$ where $u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$; BC: $u(0,t) = t \text{ and } u_x(0,t) = e^{-t}$

Order can be reduced by let $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$

$u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\} = 6xe^{-t} = \frac{\partial}{\partial x}\{p(x,t)\}$

- *Integrate the PDE with respect to variable x (Hence, variable t is constant)*

$\int \frac{\partial}{\partial x}\{p(x,t)\}dx = \int 6xe^{-t}dx$

$p(x,t) = 6e^{-t}\int x dx = 6e^{-t}\frac{x^2}{2} + c_1(t)$

- *Back substitution the $p(x,t) = \frac{\partial}{\partial x}\{u(x,t)\}$. Hence, Integrate the PDE with respect to variable x (Note: variable t is constant in this case)*

$\int \frac{\partial}{\partial x}\{u(x,t)\}dx = \int 3e^{-t}x^2 + c_1(t)\, dx$

$\therefore u(x,t) = e^{-t}x^3 + xc_1(t) + c_2(t)$