APPENDIX 3.1 CONVERSION BETWEEN EXPONENTIAL & TRIGONOMETRIC FUNCTIONS

Complete solution:

>> $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

where $m_1 \neq m_2$;

$m_1 = m + i\beta$ & $m_2 = m - i\beta$;

$i = \sqrt{-1} = imaginary$

or

>> $y(x) = e^{mx}(Acos\beta x + Bsin\beta x)$

where $A = c_1 + c_2$;

$B = i(c_1 - c_2)$

Note: In this case, the complete solution can be either $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ and $y(x) = e^{mx}(Acos\beta x + Bsin\beta x)$. Both are the same equation but in different format.

See the proof below.

$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

>> $y(x) = c_1 e^{(m+i\beta)x} + c_2 e^{(m-i\beta)x}$

$= c_1 e^{(m)x} e^{(i\beta)x} + c_2 e^{(m)x} e^{(-i\beta)x}$

$= e^{mx} \left(c_1 e^{(i\beta)x} + c_2 e^{(-i\beta)x} \right)$

Given Euler formula: $e^{ix} = cosx + isinx$; $e^{-ix} = cosx - i(sinx)$

>> $y(x) = e^{mx} \left(c_1(cos\beta x + i(sin\beta x)) + c_2(cos\beta x - i(sin\beta x)) \right)$

$= e^{mx} \left(cos\beta x(c_1 + c_2) + i(sin\beta x)(c_1 - c_2) \right)$

$= e^{mx} (Acos\beta x + Bsin\beta x)$ [proven]

Thus, exponential function, $c_1 e^{(i\beta)x} + c_2 e^{(-i\beta)x}$ can be converted to trigonometric function, $Acos\beta x + Bsin\beta x$ using the Euler formula. Example: $c_1 e^{i(5x)} + c_2 e^{-i(5x)} = Acos5x + Bsin5x$

APPENDIX 3.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH NON-CONSTANT COEFFICIENTS x^2, ax (KNOWN AS EULER-CAUCHY DIFFERENTIAL EQUATION)

In section 3.3, we discussed homogeneous linear differential equation with constant coefficient, i.e. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ and let the solution to be $y(x) = e^{mx}$ or $y(x) = xe^{mx}$ depending on the roots of characteristic equation.

However this methods is not applicable to solve the Euler-Cauchy differential equation, i.e. $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, where the coefficients are not constant. The strategy to solve this type of differential equation is to **convert the non-constant coefficient into constant form**. This can be achieved by substitution (let $x = e^t$).

Two important properties used to convert non-constant coefficient to constant coefficient:

(i) $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

(ii) $x\frac{dy}{dx} = \frac{dy}{dt}$

The detail description and proof is provided in the table below.

(i) Convert non-constant coefficient $\left(x\frac{dy}{dx}\right)$ to constant coefficient $\left(\frac{dy}{dt}\right)$
$x = e^t$ >> $ln\|x\| = t$ >> $\frac{1}{x} = \frac{dt}{dx}$ >> $\frac{1}{x} = \frac{dt}{dx}$ Using chain rule, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ >> $\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{1}{x}\right)$ >> $x\frac{dy}{dx} = \frac{dy}{dt}$

(ii) Convert non-constant coefficient $\left(x^2 \frac{d^2y}{dx^2}\right)$ to constant coefficient $\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$
$x\frac{dy}{dx} = \frac{dy}{dt}$ >> $\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)$ >> $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right)$ Using chain rule, $\frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)$ where $\frac{1}{x} = \frac{dt}{dx}$ >> $\frac{d}{dx}\left(\frac{dy}{dt}\right) = \left(\frac{d^2y}{dt^2}\right)\left(\frac{1}{x}\right)$
Combining the equations, we get $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \left(\frac{d^2y}{dt^2}\right)\left(\frac{1}{x}\right)$ >> $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dt^2}$ where $x\frac{dy}{dx} = \frac{dy}{dt}$ >> $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

For example: Solve $2x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 3y = 0.$

Solution: Let $x = e^t$, then we get

(i) $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

(ii) $x\frac{dy}{dx} = \frac{dy}{dt}$

$2x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 3y = 0$ [*Euler-Cauchy differential equation,* $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$]

>> $2\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3\left(\frac{dy}{dt}\right) - 3y = 0$

>> $2\left(\frac{d^2y}{dt^2}\right) - 5\left(\frac{dy}{dt}\right) - 3y = 0$ [*2nd order linear homogeneous DE with Constant coefficient*]

>> $2(m^2) - 5(m) - 3 = 0$ [*Characteristic equation*]

where

$b^2 - 4ac = (-5)^2 - 4(2)(-3) = 49 > 0$, thus it is Case (a) $m_1 \neq m_2$

>> $(2m+1)(m-3) = 0$

>>$m_1 = -0.5, m_2 = 3$ [*Real and distinct roots case*]

The complete solution: $y(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

>> $y(t) = c_1 e^{-0.5t} + c_2 e^{3t}$

Back substitution, we get the complementary solution to the $2x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 3y = 0$.

>> $x = e^t$

>> $y(x) = c_1 x^{-0.5} + c_2 x^3$ where c_1 & $c_2 = arbitrary\ constants$.

APPENDIX 5.1 MATHEMATIC MODELLING OF A LIQUID SYSTEM (1ST ORDER ODE)

Figure A5.1 shows a tank of liquid. The tank has a constant cross-sectional area *A*. The liquid can flow out of the tank through a valve near the base. As it does, the height or head, *h*, of liquid in the tank will reduce. Let *q* be the rate at which liquid flows out of the tank. Under certain conditions the rate outflow is proportional to the head, so that *q* = *kh* where *k* is a constant of proportionality. Situation like this arises frequently in chemical engineering industry.

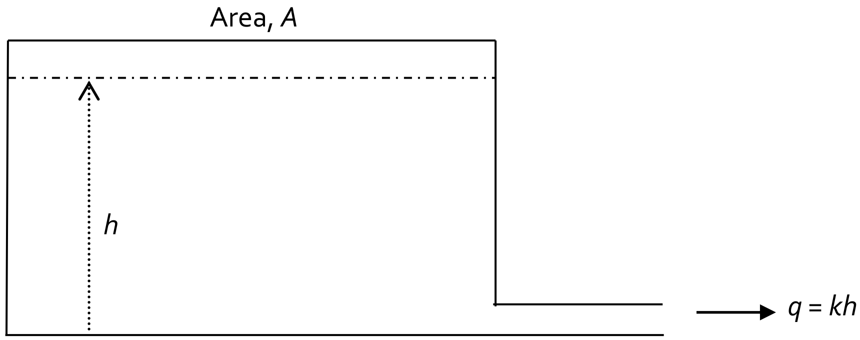


Figure A5.1. Modelling a liquid system

Mathematic modelling of the liquid system above is illustrated below:

The expression for the volume V of liquid in the tank at any time.

$$V = A \times h$$

The volume of liquid in the tank changes because liquid is flowing out.

Based on the law of conservation of mass:

The rate at which this volume changes = rate of flow in – rate of flow out

$$\frac{dV}{dt} = 0 - q = -q$$

Note: Rate of flow in is zero because there is no flow into the tank.

Since $V = Ah$ where A is constant, so the rate of change of volume can be further reduced to

$$\frac{dV}{dt} = \frac{d(Ah)}{dt} = A\frac{dh}{dt} = -q$$

Also, $q = kh$, where k= arbitrary constant, so

$$A\frac{dh}{dt} = -kh \qquad \text{or} \qquad Ah' + kh = 0$$

This is a first order differential equation with dependent variable *h* and independent variable *t*. It is linear and has constant coefficients. The unknown function to seek (the solution) is *h*(*t*). Engineer will always solve the equation to find the head, *h*, at any time, *t* for designing a liquid system.

APPENDIX 5.2 MATHEMATIC MODELLING OF AN RLC ELECTRICAL CIRCUIT (2^{ND} *ORDER ODE)*

Figure A5.2 shows an *RLC* circuit. This is a circuit comprising an inductor of inductance *L*, a capacitor of capacitance *C*, and a resistor of resistance *R* placed in series.

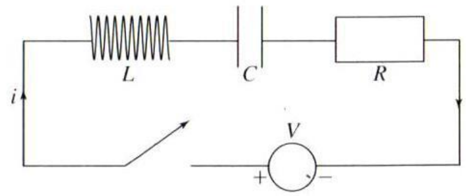


Figure A5.2. Modelling an *RLC* circuit

When a constant voltage source, *V*, is applied, it can be shown that the charge, *Q(t)*, and the current , *i(t)* which is the rate of change of *Q(t)* with respect to *t*, satisfy the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

or

$$L\,Q'' + R\,Q' + \frac{1}{C}Q = 0$$

This equation can be derived using Kirchhoff's voltage law, the individual laws for each component. Because *L*, *R*, and *C* are constants, this is a constant coefficient equation. It is linear and second order.

Notice that the RHS of the equation is zero indicates that the electromotive force excitation is zero. You may wonder how current flows by zero electromotive force. In fact, the current flow due to the initial condition. The unknown function to seek (the solution) is *Q*(*t*). Engineer will solve the equation to find the charge in the circuit, *Q*, at any time, *t* under initial condition, then find the current from the charge solution by the following relationship: $I = \frac{dQ}{dt}$, and design the circuit accordingly by adjusting the appropriate L, R & C. In this case, the complementary solution will be in transient form.

APPENDIX 5.3 MATHEMATIC MODELLING OF AN RLC ELECTRICAL CIRCUIT UNDER ELECTROMOTIVE FORCE EXCITATION (2ND ORDER ODE)

Analyze the electric circuit shown in Figure A5.3a. It contains an electromotive force *E* (supplied by a battery or generator), a resistor *R*, an inductor *L*, and a capacitor *C*, in series.

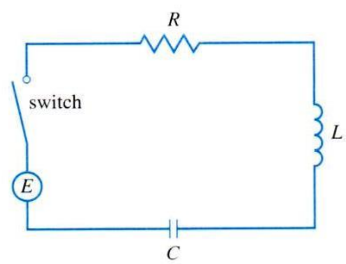


Figure A5.3a. Modelling of electric circuits

If the charge on the capacitor at time *t* is *Q* = *Q*(*t*), then the current, *I*, is the rate of change of *Q* with respect to *t*,

$$I = \frac{dQ}{dt}$$

It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively.

Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = \frac{dQ}{dt}$,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

which is a second order linear differential equation with constants coefficients.

Notice that the RHS of the equation is $E(t)$ indicates that the electromotive force excitation is non-zero. Thus, the electromotive force causes current to flow in the circuit. The unknown function to seek (the solution) is *Q*(*t*). Engineer will solve the equation to find the charge in the circuit, *i*, at any time, *t* under the electromotive force. In this case, the particular solution will be in steady state form in most of the time depending the type of electromotive force. Note that the current can be obtained from the charge solution by the following relationship: $I = \frac{dQ}{dt}$.

If the charge Q_0 and the current I_0 are known at time 0, then the initial conditions are

$$Q(0) = Q_0 \qquad\qquad Q' = I(0) = I_0$$

The total solution will be the combination of the particular solution due to the electromotive force and the complementary solution due to the initial condition.

Example

Find the charge and current at time t in the circuit of Figure A12.6a if R = 40 Ω, L = 1 H, C = 16 × 10^{-4} F, $E(t)$ = 100 cos 10t, and the initial charge and current are both 0.

Solution

With the given values of L, R, C, and $E(t)$, we obtain

$$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t \qquad , I(0) = Q(0) = 0$$

The characteristic/ auxiliary equation is $r^2 + 40r + 625 = 0$ with roots

$$r = \frac{-40 \pm \sqrt{-900}}{2} = -20 \pm 15i$$

The solution of the complementary equation is

$$Q_c(t) = e^{-20t}(c_1e^{15it} + c_2e^{-15it})$$

or

$$Q_c(t) = e^{-20t}(c_1\cos 15t + c_2\sin 15t)$$

For the method of undetermined coefficients, the particular solution is

$$Q_p(t) = A\cos 10t + B\sin 10t$$

Note: No treatment is needed as the exponential coefficient is different with the characteristic roots.

Then

$$Q_p'(t) = -10A\sin 10t + 10B\cos 10t$$

$$Q_p''(t) = -100A\cos 10t - 100B\sin 10t$$

Substituting into Equation

$$(525A + 400B)\cos 10t \ + (-400A + 525B)\sin 10t = 100\cos 10t$$

Equating coefficients,

$$525A + 400B = 100 \qquad\qquad -400A + 525B = 0$$

or

$$21A + 16B = 4 \qquad\qquad -16A + 21B = 0$$

The solution is

$$A = \frac{84}{697} \qquad\qquad B = \frac{64}{697}$$

and the particular solution is

$$Q_p(t) = \frac{1}{697}(84\cos 10t + 64\sin 10t)$$

The general solution is

$$Q(t) = Q_c(t) + Q_p(t) = e^{-20t}(c_1\cos 15t + c_2\sin 15t) + \frac{4}{697}(21\cos 10t + 16\sin 10t)$$

Imposing the initial condition $Q(0) = 0$

$$Q(0) = c_1 + \frac{84}{697} = 0 \qquad\qquad \text{or} \qquad\qquad c_1 = -\frac{84}{697}$$

To impose the other initial condition, first differentiate to find the current

$$I = \frac{dQ}{dt} = e^{-20t}[(-20c_1 + 15c_2)\cos 15t + (-15c_1 - 20c_2)\sin 15t] + \frac{40}{697}(-21\sin 10t + 16\cos 10t)$$

$$I(0) = -20c_1 + 15c_2 + \frac{640}{697} = 0 \qquad\qquad \text{or} \qquad\qquad c_2 = -\frac{464}{2091}$$

Thus, the formula for charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63\cos 15t - 116\sin 15t) + (21\cos 10t + 16\sin 10t)\right]$$

and the expression for the current is

$$I(t) = \frac{1}{2091}\left[e^{-20t}(-1920\cos 15t + 13.060\sin 15t) + 120(-21\sin 10t + 16\cos 10t)\right]$$

Note 1:

The solution for Q(t) consists of two parts. Since $e^{-20t} \to 0$ as $t \to \infty$ and both cos 15t and sin 15t are bounded functions

$$Q_c(t) = \frac{4}{2091}e^{-20t}(-63\cos 15t - 116\sin 15t) \to 0 \quad \text{as} \quad t \to \infty$$

So, for large values of t, the total solution is approximate to the particular solution

$$Q(t) \approx Q_p(t) = \frac{4}{697}(21\cos 10t + 16\sin 10t)$$

and, for this reason, $Q_p(t)$ is called the steady-state solution. Figure A5.3b shows how the graph of the steady state behavior of particular solution, Q_p compares with total solution, Q.

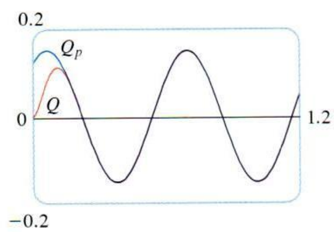
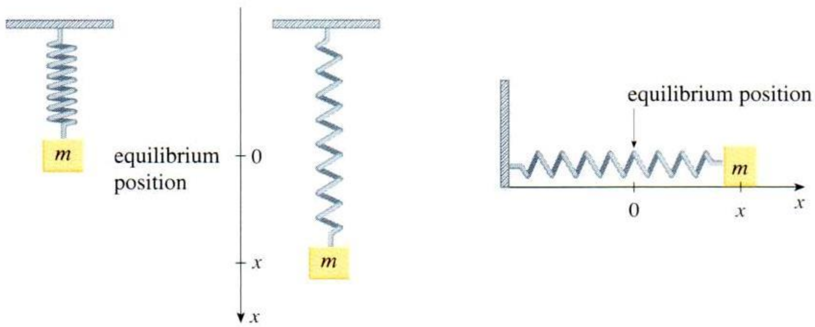


Figure A5.3b. Graphs of $Q(t)$ and $Q_p(t)$

APPENDIX 5.4 MATHEMATIC MODELLING OF A VIBRATING SPRING WITHOUT DAMPING (2^{ND} *ORDER ODE)*

Consider the motion of an object with mass m at the end of the spring that is either vertical or horizontal on a level surface as shown in Figure A5.4.



(a) (b)

Figure A5.4. Modelling of vibrating spring (a) vertical (b) horizontal

Using Hooke's Law, which says that if the spring is stretched (or compressed) *x* units from its natural length, then it exerts a force that is proportional to *x*

$$\text{restoring force} = -kx$$

where *k* is a positive constant (called the spring constant).

If any external resisting forces (due to air resistance or friction) are ignored, then by Newton's Second Law, *F* = *ma*,

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$m\frac{d^2x}{dt^2} + kx = 0$$

This is a second order linear differential equation.

Solution:

The characteristic/auxiliary equation is

$$mr^2 + k = 0$$

with roots $r = \pm\omega i$ where $\omega = \sqrt{\frac{k}{m}}$

Thus, the general solution is

$$x(t) = c_1 e^{\omega it} + c_1 e^{-\omega it} = c_1 \cos \omega t + \ c_2 \sin \omega t = A \cos(\omega t + \ \delta)$$

where

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(Natural Frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\delta = \cos^{-1}\left(\frac{c_1}{A}\right) \ = \sin^{-1}\left(-\frac{c_2}{A}\right) = \sin^{-1}\left(\frac{c_2}{c_1}\right) \qquad \text{(phase angle)}$$

This type of motion is called simple harmonic motion. Note that the natural frequency is an important characteristic of a vibrating structure.

Example 1

A spring with mass of 2kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time *t*.

Solution

From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

$$k = \frac{25.6}{0.2} = 128$$

Using this value of the spring constant *k*, together with *m* = 2 in Equation 1

$$2\frac{d^2x}{dt^2} + 128x = 0$$

The solution of this equation is

$$x(t) = c_1 \cos 8t + \ c_2 \sin 8t \qquad (2)$$

The initial condition is given as $x(0) = 0.2$. Hence from Equation (2), $x(0) = c_1 = 0.2$.

Differentiating Equation (2)

$$x'(t) = -8c_1 \cos 8t + \ 8c_2 \sin 8t$$

Since the initial velocity is given as $x'(0) = 0$, $c_2 = 0$

So, the solution is $x(t) = 0.2 \cos 8t$

Comment: Without damping, the vibration will be continue forever without any energy loss

APPENDIX 5.5 MATHEMATIC MODELLING OF A VIBRATING SPRING WITH DAMPING (2^{ND} ORDER ODE)

Consider the motion of a spring that is subject to a frictional force (in the case of horizontal spring) or a damping force (in the case where a vertical spring moves through a fluid), as shown in shown in Figure A5.5a. An example is the damping force supplied by a shock absorber in a car or a bicycle.

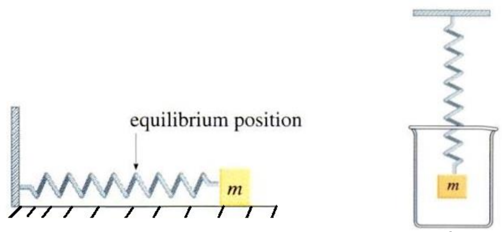


Figure A5.5a. Modelling of (a) horizontal spring on a frictional floor & (b) vertical spring in a fluid

Assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Thus,

$$\text{damping force} = -c\frac{dx}{dt}$$

where *c* is a positive constant, called the damping constant.

Thus, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This is a second order linear differential equation.

Solution:

The auxiliary equation is

$$mr^2 + cr + k = 0$$

with roots

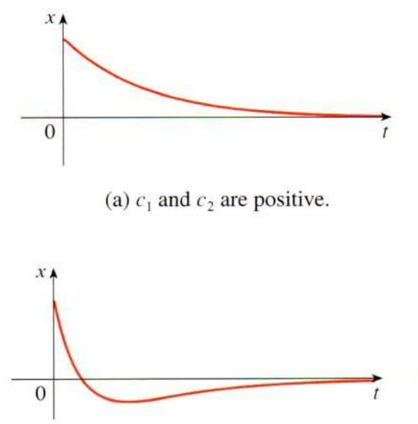
$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad\qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

Case I: $c^2 - 4mk > 0$ (***overdamping***)

The roots, r_1 and r_2, are distinct real roots and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since c, m, and k are all positive, then $\sqrt{c^2 - 4mk} < 0$, and the obtained roots r_1 and r_2 must be both negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of t are shown in Figure A5.5b. Notice oscillations do not occur. This is because $c^2 > 4mk$ means that there is a strong damping force (high viscosity oil or grease) compared with a weak spring or small mass.



(a) c_1 and c_2 are positive.

(b) c_1 and c_2 have opposite signs.

Figure A5.5b. Typical graphs for overdamping case

Case II: $c^2 - 4mk = 0$ (***critical damping***)

This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

The solution is given by

$$x = (c_1 + c_2 t)e^{-\left(\frac{c}{2m}\right)t}$$

and a typical graph is shown in Figure A5.5c. It is similar to Case I, but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of fluid leads to the vibrations of the following case.

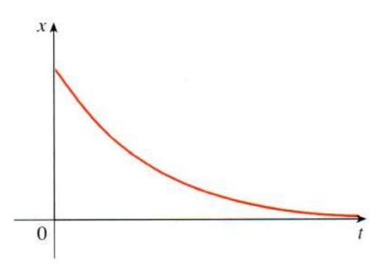


Figure A5.5c. Graph for critical damping case

Case III: $\boldsymbol{c^2 - 4mk < 0}$ (***underdamping***)

Here the roots are complex

$$r_{1,2} = -\frac{c}{2m} \pm \omega i$$

where

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

The solution is given by

$$x = e^{-\left(\frac{c}{2m}\right)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

There are oscillations that are damped by the factor $e^{-\left(\frac{c}{2m}\right)t}$. Since $c > 0$ and $m > 0$, then $e^{-\left(\frac{c}{2m}\right)t} < 0$ so $e^{-\left(\frac{c}{2m}\right)t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure A5.5d

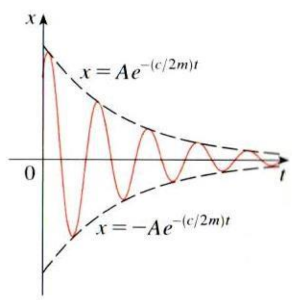


Figure A5.5d. Graph for underdamping case

Note: In actual engineering problem, most of the vibration system is in under-damped case with oscillation. Damping is controlled or designed appropriately so that oscillation of the vibration can be reduced significantly such as the one in critical-damped case.

<u>Example 2</u>

Suppose that the spring in **<u>Example 1</u>** is immersed in a fluid with damping constant *c* = 40. Find the position of the mass at any time *t* if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

<u>Solution</u>

The mass is *m* = 2 and the spring constant is *k* = 128, so the differential equation becomes

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

or

$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

The auxiliary equation is

$$r^2 + 20r + 64 = (r + 4)(r + 16) = 0$$

with roots -4 and -16, so the motion is overdamped and the solution is

$$x(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

Given $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating,

$$x'(t) = -4c_1 e^{-4t} - 16c_2 e^{-16t}$$

So

$$x'(0) = -4c_1 - 16c_2 = 0.6$$

Since $c_1 = -c_2$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$. Therefore

$$x(t) = 0.05(e^{-4t} - e^{-16t})$$

Comment: With damping, the vibration will be reduced over the time due to energy loss

APPENDIX 5.6 MATHEMATIC MODELLING OF A DAMPED VIBRATION SYSTEM UNDER FORCED VIBRATION (2ND ORDER ODE)

Suppose that, in addition to the restoring force and the damping force, the motion of the mass is affected by an external force *F*(*t*).

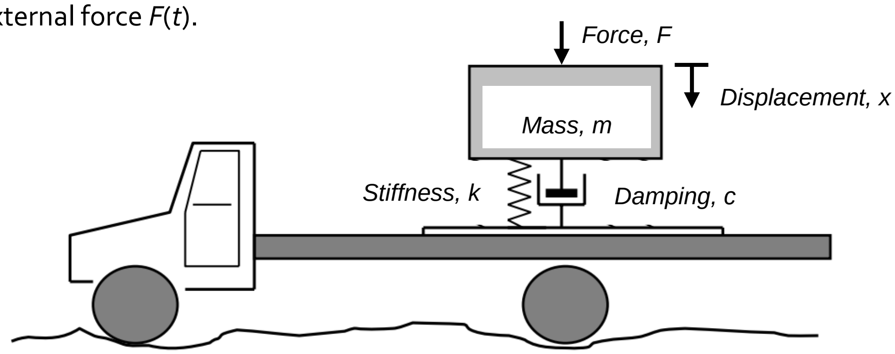


Figure A5.6. Example of a vibrating damped mass-spring system due to road excitation

The Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} + \text{external force} = -kx - c\frac{dx}{dt} + F(t)$$

Thus, the motion of the spring is now governed by the following nonhomogeneous differential equation.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \qquad (5)$$

A commonly occurring type of external force is a periodic forcing function

$$F(t) = F_0 \cos \omega_0 t$$

where $\omega_0 = exciting\ frequency\ of\ the\ forcing\ function$

$F_0 = amplitude\ of\ the\ force$

In this case, and in the absence of a damping force (c = 0), we can obtain the solution to be

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t \qquad (6)$$

If $\omega = \omega_0$, then the applied frequency, ω_0 reinforces the natural frequency, $\omega = \sqrt{\frac{k}{m}}$ and the result is vibrations of large amplitude. This is the phenomenon of resonance which is unfavorable in many engineering problems. Engineers should understand the natural frequency and exciting frequency of a structure to design it appropriately.

APPENDIX 5.7 ANALOGUE BETWEEN VIBRATIONAL SYSTEM AND ELECTRIC CIRCUIT

Some engineering systems, like the mechanical mass-spring-damper system and the electrical RLC circuit have similarities. These similarities can be seen in control system or control engineering study. When comparing, they are often analogous to each other. A common analogue is the force-voltage analogue as shown in Table A5.7.

Table A5.7. Physical entities versus mathematical entities

Mass Spring Damper System		Electrical RLC Circuit	
x	Displacement	Q	Charge
$\frac{dx}{dt}$	Velocity	$I = \frac{dQ}{dt}$	Current
m	Mass	L	Inductance
c	Damping constant	R	Resistance
k	Spring constant (i.e stiffness)	$\frac{1}{C}$	Elastance (i.e. inverse of capacitance)
$F(t)$	External force	$E(t)$	Electromotive force