FOURIER SERIES EXPANSION & ITS APPLICATION

WEEK 11: FOURIER SERIES EXPANSION & ITS APPLICATION

11.1 INTRODUCTION

In the previous chapter, we learn how to convert the periodic time signal into Fourier series expression.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx;$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos n\omega x \; dx;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin n\omega x \, dx \,;$$

$$f(x) = f(x + np) \;\; for \quad -\infty \le x \le \infty$$

Comment: Fourier series only works for periodic signal that repeats itself in all the time or within an infinite time interval. However, in actual engineering practice, we are not able to measure a periodical signal within an infinite interval, i.e. $-\infty \le x \le \infty$.

For example, we only able to measure the vibration of a machine due to motor excitation in a finite interval, i.e. measurements starts from $0 \le t \le \tau$ where *τ is the finite interval* as shown in the Figure 11.1. In this case, the measured vibration is a non-periodic signal because it does not repeat itself within infinite interval.

Thus, we cannot find Fourier series for the measured signal because the compulsory condition of the periodic signal, where $-\infty \le t \le \infty$ is not valid in this case. To solve this problem, we need to *expand or extend the finite interval to infinite interval* with "Fourier series expansion".

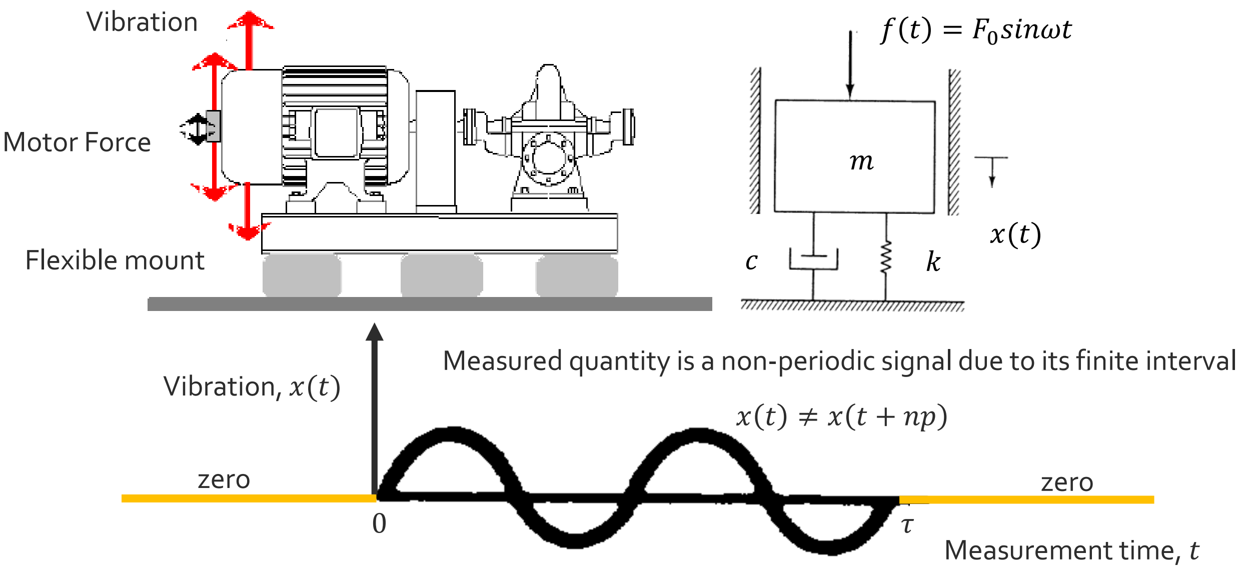
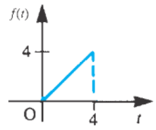


Figure 11.1: Actual vibration data obtained from measurement has finite measurement time, τ.

11.2 TYPES OF FOURIER SERIES EXPANSION

In simple, Fourier series expansion is a technique used to *convert a non-periodic signal to periodic signal* through "expansion" technique, so that a non-periodic signal can be written in the Fourier series expression. This can be done by *assuming the signal repeats itself within infinite interval*. In this way, the finite interval can be expanded/ extended to infinite interval.



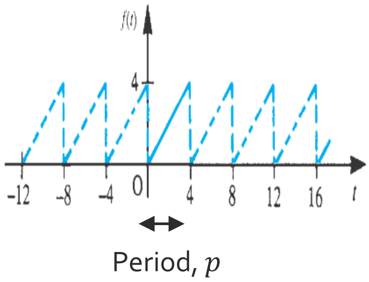
$f(t)\ is\ within\ 0 \leq t \leq \tau$

There are three types of expansion to represent non-periodic signal in the Fourier series expression:

(i) Full-range Fourier Series Expansion
-also known as Even & Odd functions expansion

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

	Conventional approach	*Alternative approach*
1	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$	$= \frac{1}{2L}\int_{0}^{\tau} f(t)\, dt$
2	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ dt$	$= \frac{1}{L}\int_{0}^{\tau} f(t) \cos n\omega t\ dt$
3	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$	$= \frac{1}{L}\int_{0}^{\tau} f(t) \sin n\omega t\, dt$

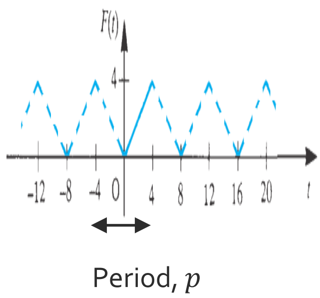


Hint: Both approaches give the same answer, but alternative approach can compute faster. The derivation of the formula can be found in Appendix 11.1.

(ii) Half-range Fourier Cosine Series Expansion
-also known as Even function expansion

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t)$$

	Conventional approach	*Alternative approach*
1	$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt$	$= \frac{1}{L}\int_{0}^{\tau} f(t)\, dt$
2	$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ dt$	$= \frac{2}{L}\int_{0}^{\tau} f(t) \cos n\omega t\ dt$

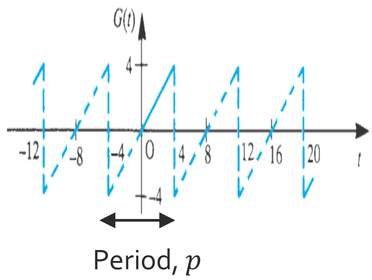


Hint: The derivation of the formula can be found in Appendix 11.1.

(iii) Half-range Fourier Sine Series Expansion
-also known as Odd functions expansion

$$f(t) = \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

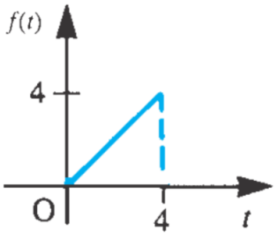
	Conventional approach	*Alternative approach*
1	$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt$	$= \frac{2}{L}\int_{0}^{\tau} f(t) \sin n\omega t\, dt$



Hint: The derivation of the formula can be found in Appendix 11.1.

Note: Half range series is widely applied because less coefficients are required to be computed.

Example 1: Find the Fourier Series for the following signal.



Observation:

(i) The $f(t)$ is valid for certain interval $0 \leq t \leq \tau$ *only, where the finite interval,* $\tau = 4$.

(ii) The $f(t)$ is a non-periodic function because $f(t) \neq f(t+np)$, $where\ n = 1,2,3, \ldots\ and\ p = period)$

Result:

Fourier series can be applied for periodic signal only using the definition below.

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt;$

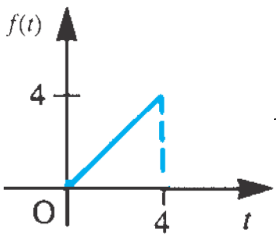
$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos n\omega t\ dt;$

$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin n\omega t\, dt\,;$

$f(t) = f(t+np)\ \ for\ \ -\infty \leq x \leq \infty$

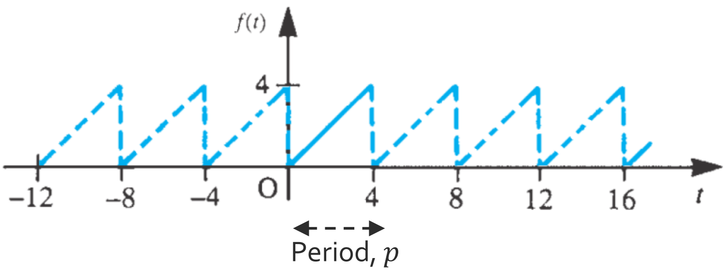
Therefore, it is not applicable in this case (i.e. Fourier series is not applicable for non-periodic signal).

Example 2: Find the Full Range Fourier Series Expansion for the following signal.



$f(t) = t\ is\ within\ 0 \leq t \leq \tau$

Step 1: Performing the Full Range Expansion (i.e. Even & Odd functions expansion)



Note: In this approach, the full range of the signal, i.e. $0 \le t \le 4$ is assumed to be repeating itself within infinite interval (i.e. "expansion"). In this way, the non-periodic signal is converted to periodic signal, where $f(t) = f(t + 4n) \quad for \quad -\infty \le t \le \infty$

Step 2: Important Parameter of the Signal

$\tau = 4$

$p = 4, L = 2, \omega = \dfrac{2\pi}{p} = \dfrac{\pi}{2}$

Step 2: Computing the Full Range Fourier Series Expansion (Alternative approach)::

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

where $a_0 = \frac{1}{2L}\int_0^{\tau} f(t)\,dt = \frac{1}{4}\int_0^4 t\,dt = \frac{1}{4}\left[\frac{t^2}{2}\right]_0^4 = \frac{1}{4}(8) = 2;$

$$a_n = \frac{1}{L}\int_0^{\tau} f(t)\cos n\omega t\ \ dt = \frac{1}{2}\int_0^4 t\cos n\frac{\pi}{2}t\,dt;$$

$$Integration\ by\ part: Let\ u = t; dv = \cos n\frac{\pi}{2}t\,dt$$

$$a_n = \left[t(\frac{\sin\left(n\frac{\pi}{2}t\right)}{n\frac{\pi}{2}}\right]_0^4 - \int_0^4 \frac{\sin\left(n\frac{\pi}{2}t\right)}{n\frac{\pi}{2}}dt$$

$$= 0 - \left[(\frac{-\cos\left(n\frac{\pi}{2}t\right)}{\left(n\frac{\pi}{2}\right)^2}\right]_0^4$$

$$= (\frac{\cos(2n\pi)}{\left(n\frac{\pi}{2}\right)^2} - \frac{1}{\left(n\frac{\pi}{2}\right)^2} = 0$$

Hint: $\cos(2n\pi) = 1$

$$b_n = \frac{1}{L}\int_0^{\tau} f(t)\sin n\omega t\,dt = \frac{1}{2}\int_0^4 t\sin n\frac{\pi}{2}t\,dt\ ;$$

$$Integration\ by\ part: Let\ u = t; dv = \sin n\frac{\pi}{2}t\,dt$$

$$= \frac{1}{2}\left\{\left[t(\frac{-\cos\left(n\frac{\pi}{2}t\right)}{n\frac{\pi}{2}}\right]_0^4 - \int_0^4 \frac{-\cos\left(n\frac{\pi}{2}t\right)}{n\frac{\pi}{2}}dt\right\}$$

$$= \frac{1}{2}\left\{4\left(\frac{-\cos(2n\pi)}{n\frac{\pi}{2}}\right) - \left[(\frac{-\sin\left(n\frac{\pi}{2}t\right)}{\left(n\frac{\pi}{2}\right)^2}\right]_0^4\right\}$$

Hint: $\sin(2n\pi) = 0$

$$= \frac{1}{2}\left\{\frac{-8}{n\pi}\ \right\} = \frac{-4}{n\pi}$$

Step 3: Final Solution

$$\therefore f(t) = 2 - \sum_{n=1}^{\infty}\left(0 + \frac{-4}{n\pi} sin\, n\omega t\right)$$
$$= 2 - \frac{4}{\pi}\left(sin\frac{\pi}{2}t + \frac{1}{2}sin\pi t + \frac{1}{3}sin\frac{3\pi}{2}t + \cdots\right)$$

$f(t)$ *is valid only for* $0 \le t \le 4$

Extra info: It is not recommended to use the conventional approach to determine the Fourier series expansion as alternative approach can compute it much faster than the conventional approach. This is illustrated in the example below:

Step 2: Computing the Full Range Fourier Series Expansion (Conventional approach):

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$where\ a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt = \frac{1}{4}\int_{-2}^{2} f(t)dt$$
$$= \frac{1}{4}\left(\int_{-2}^{0}(t+4)dt + \int_{0}^{2} tdt\right) = \frac{1}{4}\left(\left[\frac{t^2}{2} + 4t\right]_{-2}^{0} + \left[\frac{t^2}{2}\right]_{0}^{2}\right) = \frac{1}{4}(6 + (2)) = 2;$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\, cos\, n\omega t\ dt = \frac{1}{2}\int_{-2}^{2} f(t)\, cos\, n\frac{\pi}{2}t\, dt;$$
$$= \frac{1}{2}\left(\int_{-2}^{0}(t+4)\, cos\, n\frac{\pi}{2}t\, dt + \int_{0}^{2} t\, cos\, n\frac{\pi}{2}t\, dt\right)$$
$$= \cdots \textit{two integration by part are not shown here for simplification} \ldots;$$
$$= \frac{2}{n\pi}sin(n\pi) - \frac{2}{n\pi}sin(-n\pi) + \frac{2}{n^2\pi^2}cos(n\pi) - \frac{2}{n^2\pi^2}cos(-n\pi);$$
$$= \frac{4}{n\pi}sin(n\pi);$$
Hint: $sin(n\pi) = 0$
$$= 0;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\, sin\, n\omega t\, dt = \frac{1}{2}\int_{-2}^{2} f(t)\, sin\ n\frac{\pi}{2}t\, dt\,;$$
$$= \frac{1}{2}\left(\int_{-2}^{0}(t+4)\, sin\, n\frac{\pi}{2}t\, dt + \int_{0}^{2} t\, sin\, n\frac{\pi}{2}t\, dt\right)$$
$$= \ldots \textit{two integration by part are not shown here for simplification} \ldots;$$
$$= \frac{2}{n\pi}cos(-n\pi) - \frac{2}{n\pi}cos(n\pi) + \frac{2}{n^2\pi^2}sin(n\pi) - \frac{2}{n^2\pi^2}sin(-n\pi) - \frac{4}{n\pi};$$
$$= \frac{4}{n^2\pi^2}sin(n\pi) - \frac{4}{n\pi}$$
$$= -\frac{4}{n\pi}$$

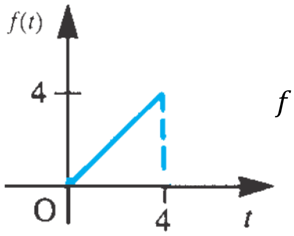
Step 3: Final Solution

$$\therefore f(t) = 2 - \sum_{n=1}^{\infty}\left(0 + \frac{-4}{n\pi} sin\, n\omega t\right)$$
$$= 2 - \frac{4}{\pi}\left(sin\frac{\pi}{2}t + \frac{1}{2}sin\pi t + \frac{1}{3}sin\frac{3\pi}{2}t + \cdots\right)$$

$f(t)$ *is valid only for* $0 \le t \le 4$

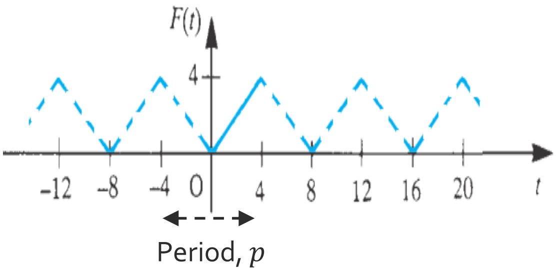
Comment: Produce the same answer like the previous answer but the steps are longer.

Example 3: Find the Half Range Fourier Cosine Series Expansion for the signal.



$f(t) = t \text{ is within } 0 \leq t \leq \tau$

Step 1: Performing the Half Range Cosine Series Expansion (i.e. Even function expansion)



Note: In this approach, the half range of the signal, i.e. $0 \leq t \leq 4$ and half range of its mirror are assumed to be repeating itself within infinite interval (i.e. "expansion"). In this way, the non-periodic signal is converted to periodic signal, where $f(t) = f(t + 8n) \quad for \quad -\infty \leq t \leq \infty$

Step 2: Important Parameter of the Signal

$\tau = 4$

$p = 8, L = 4, \omega = \dfrac{2\pi}{p} = \dfrac{\pi}{4}$

Step 2: Computing the Half Range Cosine Series Expansion (Alternative approach)::

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t)$$

where $a_0 = \frac{1}{L}\int_0^{\tau} f(t)\, dt = \frac{1}{4}\int_0^4 t\, dt = \frac{1}{4}\left[\frac{t^2}{2}\right]_0^4 = \frac{1}{4}(8) = 2;$

$$a_n = \frac{2}{L}\int_0^{\tau} f(t) \cos n\omega t \; dt = \frac{2}{4}\int_0^4 t \cos n\frac{\pi}{4} t\, dt;$$

$$Integration\ by\ part: Let\ u = t; dv = \cos n\frac{\pi}{4} t\, dt$$

$$a_n = \frac{1}{2}\left\{\left[t\left(\frac{\sin\left(n\frac{\pi}{4}t\right)}{n\frac{\pi}{4}}\right)\right]_0^4 - \int_0^4 \frac{\sin\left(n\frac{\pi}{4}t\right)}{n\frac{\pi}{4}} dt\right\}$$

$$= \frac{1}{2}\left\{0 - \left[\left(\frac{-\cos\left(n\frac{\pi}{4}t\right)}{\left(n\frac{\pi}{4}\right)^2}\right]_0^4\right\} = \frac{1}{2}\left\{\left(\frac{\cos(n\pi t)}{\left(n\frac{\pi}{4}\right)^2} - \frac{1}{\left(n\frac{\pi}{4}\right)^2}\right\}$$

$$= \frac{8}{(n\pi)^2}\{(-1)^n - 1\}$$

Hint: $\sin(n\pi) = 0$

$$\cos(n\pi) = \begin{cases} -1 & odd\ n \\ 1 & even\ n \end{cases} = (-1)^n$$

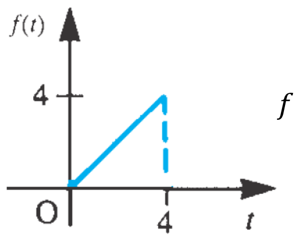
Step 3: Final Solution

$$\therefore f(t) = 2 + \sum_{n=1}^{\infty}\left(\frac{8}{(n\pi)^2}\{(-1)^n - 1\}\cos n\omega t\right)$$

$$= 2 + \frac{8}{\pi^2}\left(-2cos\frac{\pi}{4}t - \frac{2}{9}cos\frac{3\pi}{4}t - \frac{2}{25}cos\frac{5\pi}{4}t - \cdots\right)$$

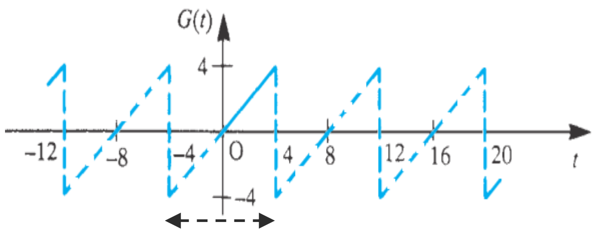
$f(t)\ is\ valid\ only\ for\ 0 \leq t \leq 4$

Example 4: Find the Half Range Fourier Sine Series Expansion for the signal.



$f(t) = t\ is\ within\ 0 \leq t \leq \tau$

Step 1: Performing the Half Range Sine Series Expansion (i.e. Odd functions expansion)



Note: In this approach, the half range of the signal, i.e. $0 \leq t \leq 4$ and half range of its upside down mirror are assumed to be repeating itself within infinite interval (i.e. "expansion"). In this way, the non-periodic signal is converted to periodic signal, where $f(t) = f(t + 8n)\quad for - \infty \leq t \leq \infty$.

Step 2: Important Parameter of the Signal

$\tau = 4$

$p = 8, L = 4, \omega = \frac{2\pi}{p} = \frac{\pi}{4}$

Step 2: Computing the Half Range Sine Series Expansion (Alternative approach)::

$$f(t) = \sum_{n=1}^{\infty}(b_n \sin n\omega t)$$

where $b_n = \frac{2}{L}\int_0^{\tau} f(t)\, sin\, n\omega t\, dt = \frac{2}{4}\int_0^4 t\, sin\, n\frac{\pi}{4}t\, dt;$

$$Integration\ by\ part: Let\ u = t; dv = sin\, n\frac{\pi}{4}t\, dt$$

$$b_n = \frac{1}{2}\left\{\left[t\left(\frac{-cos\left(n\frac{\pi}{4}t\right)}{n\frac{\pi}{4}}\right]_0^4 - \int_0^4 \frac{-cos\left(n\frac{\pi}{4}t\right)}{n\frac{\pi}{4}}dt\right\}$$

$$= \frac{1}{2}\left\{4\left(\frac{-cos(n\pi)}{n\frac{\pi}{4}} - 0\right) - \left[\left(\frac{-sin\left(n\frac{\pi}{4}t\right)}{\left(n\frac{\pi}{4}\right)^2}\right]_0^4\right\}$$

$$= \frac{1}{2}\left\{4\left(\frac{-(-1)^n}{n\frac{\pi}{4}}\right) - 0\right\}$$

$$= 2\left\{\frac{(-1)^{n+1}}{n\frac{\pi}{4}}\right\} = \frac{8}{n\pi}\{(-1)^{n+1}\}$$

Hint: $sin(n\pi) = 0$

$$cos(n\pi) = \begin{cases} -1 & odd\ n \\ 1 & even\ n \end{cases} = (-1)^n$$

Step 3: Final Solution

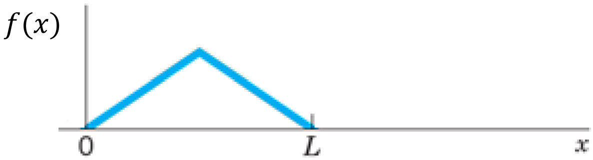
$$\therefore f(t) = \sum_{n=1}^{\infty}\left(\frac{8}{n\pi}\{(-1)^{n+1}\}\, sin\, n\omega t\right)$$

$$= \frac{8}{\pi}\left(sin\frac{\pi}{4}t - \frac{1}{2}sin\frac{2\pi}{4}t + \frac{1}{3}sin\frac{3\pi}{4}t - \cdots\right)$$

$$f(t)\ is\ valid\ only\ for\ 0 \le t \le 4$$

Example 5:

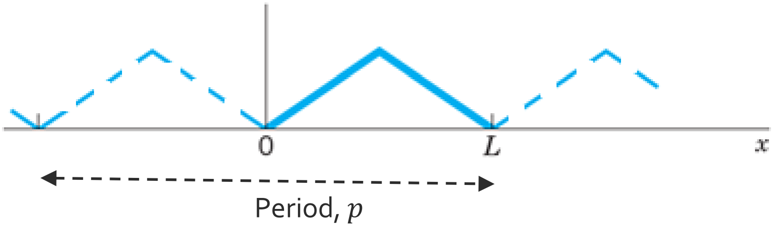
Given $f(x)$ to be the shape of a distorted violin string for $0 \leq x \leq length, L$, where



$$f(x) = \begin{cases} \frac{2k}{L}x, & 0 \leq x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} \leq x \leq L \end{cases}$$

(i) **Find the even periodic extension/ expansion** (also known as Half-Range Fourier Cosine Series expansion) to represent the deflected shape of the violin string.

Step 1: Performing the Half Range Cosine Series Expansion (i.e. Even functions expansion)



Note: In this approach, the half range of the signal, i.e. $0 \leq x \leq Length, L$ and half range of its mirror are assumed to be repeating itself within infinite interval (i.e. "expansion"). In this way, the non-periodic signal is converted to periodic signal, where $f(x) = f(x + (2\,x\,Length, L)n) \quad for\ \infty \leq t \leq \infty$

Step 2: Important Parameter of the Signal

$\tau = Length, L$

$p = 2\,x\,Length, L, Half\ of\ Period, L = Length, L, \omega = \dfrac{2\pi}{p} = \dfrac{\pi}{L}$

Step 2: Computing the Half Range Cosine Series Expansion (Alternative approach)::

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t)$$

where $a_0 = \frac{1}{L}\int_0^{\tau} f(t)\,dt = \frac{1}{L}\left[\int_0^{\frac{L}{2}} \frac{2k}{L}x dx + \int_{\frac{L}{2}}^{L} \frac{2k}{L}(L-x)dx\right]$

$$= \frac{1}{L}\left[\int_0^{\frac{L}{2}} \frac{2k}{L}x dx + \int_{\frac{L}{2}}^{L} \frac{2k}{L}(L-x)dx\right]$$

$$= \frac{1}{L}\left[\left[\frac{2k}{L}\frac{x^2}{2}\right]_0^{\frac{L}{2}} + \left[\frac{2k}{L}(Lx - \frac{x^2}{2})\right]_{\frac{L}{2}}^{L}\right]$$

$$= \frac{1}{L}\frac{2k}{L}\frac{\left(\frac{L}{2}\right)^2}{2} + \frac{1}{L}\frac{2k}{L}(L^2 - \frac{L^2}{2} - \left(\frac{L^2}{2} - \frac{\left(\frac{L}{2}\right)^2}{2}\right))$$

$$= \frac{k}{4} + \frac{k}{4} = \frac{k}{2}$$

$$a_n = \frac{2}{L}\int_0^{\tau} f(t)\cos n\omega t \; dt$$

$$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2k}{L} x \cos n\frac{\pi}{L}x\, dx + \int_{\frac{L}{2}}^{L} \frac{2k}{L}(L-x)\cos n\frac{\pi}{L}x\, dx\right);$$

$$Integration\ by\ part: Let\ u = x; dv = \cos n\frac{\pi}{L}x\, dx$$

$$\int x\cos n\frac{\pi}{L}x\, dx = uv - \int v du = x\left(\frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right) - \int \frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}dx$$

$$= \frac{xL}{n\pi}\sin\left(n\frac{\pi}{L}x\right) + \frac{\cos\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}$$

$$a_n = \frac{4k}{L^2}\left(\int_0^{\frac{L}{2}} x\cos n\frac{\pi}{L}x\, dx + L\int_{\frac{L}{2}}^{L} \cos n\frac{\pi}{L}x\, dx - \int_{\frac{L}{2}}^{L} x\cos n\frac{\pi}{L}x\, dx\right)$$

$$= \frac{4k}{L^2}\left(\begin{array}{c}\left[\frac{xL}{n\pi}\sin\left(n\frac{\pi}{L}x\right) + \frac{\cos\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}\right]_0^{\frac{L}{2}} + L\left[\frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_{\frac{L}{2}}^{L} \\ -\left[\frac{xL}{n\pi}\sin\left(n\frac{\pi}{L}x\right) + \frac{\cos\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}\right]_{\frac{L}{2}}^{L}\end{array}\right)$$

$$= \frac{4k}{L^2}\left(\left(\frac{L^2/2}{n\pi}\sin\left(n\frac{\pi}{2}\right) + \frac{\cos\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2} - \frac{1}{\left(n\frac{\pi}{L}\right)^2}\right) + L\left(\frac{\sin(n\pi)}{n\frac{\pi}{L}} - \frac{\sin\left(n\frac{\pi}{2}\right)}{n\frac{\pi}{L}}\right) - \left(\frac{\cos(n\pi)}{\left(n\frac{\pi}{L}\right)^2} - \left(\frac{L^2/2}{n\pi}\sin\left(n\frac{\pi}{2}\right) + \frac{\cos\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2}\right)\right)\right)$$

$$= \frac{4k}{L^2}\left(\frac{2\cos\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2} - \frac{\cos(n\pi)}{\left(n\frac{\pi}{L}\right)^2} - \frac{1}{\left(n\frac{\pi}{L}\right)^2}\right)$$

$$= \frac{4k}{(n\pi)^2}\left(2\cos\left(n\frac{\pi}{2}\right) - \cos(n\pi) - 1\right)$$

Step 3: Final Solution

$$\therefore f(t) = \frac{k}{2} + \sum_{n=1}^{\infty}\left(\frac{4k}{(n\pi)^2}\left(2\cos\left(n\frac{\pi}{2}\right) - \cos(n\pi) - 1\right)\cos n\frac{\pi}{L}x\right)$$

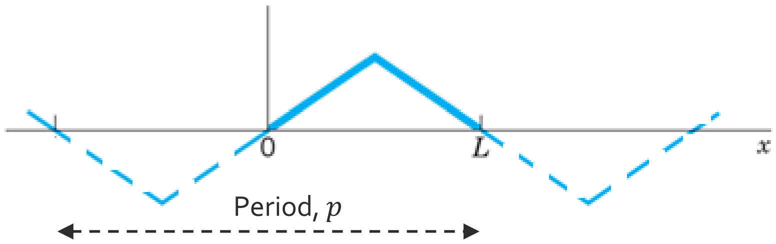
$$= \frac{k}{2} + \frac{4k}{(\pi)^2}\left(\frac{1}{(2)^2}(-4)\cos 2\frac{\pi}{L}x + \frac{1}{(6)^2}(-4)\cos 6\frac{\pi}{L}x + \frac{1}{(10)^2}(-4)\cos 10\frac{\pi}{L}x + \cdots\right)$$

$$= \frac{k}{2} - \frac{16k}{(\pi)^2}\left(\frac{1}{4}\cos 2\frac{\pi}{L}x + \frac{1}{36}\cos 6\frac{\pi}{L}x + \frac{1}{100}\cos 10\frac{\pi}{L}x + \cdots\right)$$

$$f(t)\ is\ valid\ only\ for\ 0 \le t \le L$$

(ii) **Find the odd periodic extension/ expansion** (also known as Half-Range Fourier Sine Series expansion) to represent the shape.

Step 1: Performing the Half Range Sine Series Expansion (i.e. Odd functions expansion)



Note: In this approach, the half range of the signal, i.e. $0 \le x \le Length, L$ and half range of its upside down mirror are assumed to be repeating itself within infinite interval. In this way, the non-periodic signal is converted to periodic signal, where $f(x) = f(x + (2\,x\,Length, L)n) \quad for \;\; \infty \le t \le \infty$

Step 2: Important Parameter of the Signal

$\tau = Length, L$

$p = 2\,x\,Length, L, Half\ of\ Period, L = Length, L, \omega = \dfrac{2\pi}{p} = \dfrac{\pi}{L}$

Step 2: Computing the Half Range Sine Series Expansion (Alternative approach)::

$$f(t) = \sum_{n=1}^{\infty} (b_n \sin n\omega t)$$

$where\ b_n = \frac{2}{L}\int_0^{\tau} f(t)\, sin\, n\omega t\ dt$

$= \frac{2}{L}\left(\int_0^{\frac{L}{2}} \frac{2k}{L} x\, sin\, n\frac{\pi}{L}x\, dx + \int_{\frac{L}{2}}^{L} \frac{2k}{L}(L-x)\, sin\, n\frac{\pi}{L}x\, dx\right);$

$Integration\ by\ part: Let\ u = x; dv = sin\, n\frac{\pi}{L}x\, dx$

$\int x\, sin\, n\frac{\pi}{L}x\, dx = uv - \int v du = x\left(\frac{-cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right) - \int \frac{-cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}} dx$

$= -\frac{xL}{n\pi} cos\left(n\frac{\pi}{L}x\right) + \frac{sin\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}$

$b_n = \frac{4k}{L^2}\left(\int_0^{\frac{L}{2}} x\, sin\, n\frac{\pi}{L}x\, dx + L\int_{\frac{L}{2}}^{L} sin\, n\frac{\pi}{L}x\, dx - \int_{\frac{L}{2}}^{L} x\, sin\, n\frac{\pi}{L}x\, dx\right)$

$$= \frac{4k}{L^2}\left(\begin{array}{c}\left[-\frac{xL}{n\pi} cos\left(n\frac{\pi}{L}x\right) + \frac{sin\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}\right]_0^{\frac{L}{2}} + L\left[\frac{-cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_{\frac{L}{2}}^{L} \\ -\left[-\frac{xL}{n\pi} cos\left(n\frac{\pi}{L}x\right) + \frac{sin\left(n\frac{\pi}{L}x\right)}{\left(n\frac{\pi}{L}\right)^2}\right]_{\frac{L}{2}}^{L}\end{array}\right)$$

$$= \frac{4k}{L^2}\left(\left(\left(-\frac{L^2/2}{n\pi}\cos\left(n\frac{\pi}{2}\right)+\frac{\sin\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2}\right)+L\left(\frac{-\cos(n\pi)}{n\frac{\pi}{L}}-\frac{-\cos\left(n\frac{\pi}{2}\right)}{n\frac{\pi}{L}}\right)-\left(-\frac{L^2}{n\pi}\cos(n\pi)-\left(-\frac{L^2/2}{n\pi}\cos\left(n\frac{\pi}{2}\right)+\frac{\sin\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2}\right)\right)\right)\right)$$

$$= \frac{4k}{L^2}\left(\frac{2\sin\left(n\frac{\pi}{2}\right)}{\left(n\frac{\pi}{L}\right)^2}\right)$$

$$= \frac{8k}{(n\pi)^2}\left(\sin\left(n\frac{\pi}{2}\right)\right)$$

Step 3: Final Solution

$$\therefore f(t) = \sum_{n=1}^{\infty}\left(\frac{8k}{(n\pi)^2}\left(\sin\left(n\frac{\pi}{2}\right)\right)\sin n\frac{\pi}{L}x\right)$$
$$= \frac{8k}{(\pi)^2}\left(\sin\frac{\pi}{L}x-\frac{1}{(3)^2}\sin 3\frac{\pi}{L}x+\frac{1}{(5)^2}\sin 5\frac{\pi}{L}x-\cdots\right)$$

$f(t)$ *is valid only for* $0 \le t \le L$

11.3 APPLICATION OF FOURIER SERIES EXPANSION #1: SOLVE PDE PROBLEM

Prior to Fourier's work, no solution to the PDE problem such as heat equation was known in the general case. For example:

The PDE solution, $u(x,t)$ of the heat conduction of

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

is remained unsolvable, where

for boundary conditions of $u(0,\ t) = 0$, $u(L,\ t) = 0$

PDE solution is given:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n}\sin\left(\frac{n\pi}{L}x\right)\right)$$

where $B_{3,n}$ is unknown

Assume the initial condition is given, $u(x,\ 0) = f(x)$ for $0 < x < L$. The unknown coefficients can be obtained through the Fourier Series Expansion.

$u(x,\ 0) = f(x) = \sum_{n=1}^{\infty}\left(B_{3,n}\sin\left(\frac{n\pi}{L}x\right)\right) =$ *Half range Fourier Sine Series*

where $B_{3,n} = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\,dx$ & $\omega = \frac{\pi}{L}$ & τ=L

Note: Detail discussion and calculation will be made in the next PDE chapter.