SOLVING GENERAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION (PDE)

WEEK 12: SOLVING GENERAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION (PDE)

12.1 GENERAL SOLUTION & PARTICULAR SOLUTION OF PDE

In the differential equation chapter, you have learned how to differentiate between ODE and PDE and how to classify them in terms of the order, linearity, and homogeneity. In simple, PDE is an equation that involves partial derivatives (i.e. ∂ symbol). Recall that a linear PDE is *homogeneous* if each of its terms contains either u *or* one of its partial derivatives on LHS while RHS=0. Otherwise, it is a *non-homogeneous* PDE. In this study, we will only focus on solving the 2nd order linear homogeneous PDE problem with constant coefficients.

The general equation is given below:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + Fu = 0 \quad \text{, where } A - F \text{ are constants.}$$

(i) The general PDE solution, i.e. $u(x,t)$ in terms of unknown coefficients can be obtained by using separable of variable method.
For example: $u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,\mathrm{n}}\, cos(n\pi t)\big(sin(n\pi x)\big)$

(ii) The particular PDE solution, i.e. $u(x,t)$ in terms of known coefficients can be obtained by applying all the initial & boundary conditions, as well as the Fourier series expansion method.
For example: $u_{total}(x,t) = \sum_{n=1}^{\infty} -\frac{2n\pi sinn\pi + 4cosn\pi - 4}{n^3\pi^3} cos(n\pi t)\big(sin(n\pi x)\big)$

Notation of PDE:

Note that $\frac{\partial^2 u}{\partial x^2} \neq u''$ for $\frac{\partial u}{\partial x} \neq u'$ for PDE as it has more than 1 possibility. For example, u' can be $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial t}$ while u'' can be $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, or\ \frac{\partial^2 u}{\partial t^2}$.

Thus, instead of writing u' or u'' for PDE, there is another alternative.

(i) Derivative and second derivative of $u(x,t)$ with *respect to t*

$$u_t = \frac{\partial}{\partial t}\{u(x,t)\} \ , \ u_{tt} = \frac{\partial^2}{\partial t^2}\{u(x,t)\}$$

(ii) Derivative and second derivative of $u(x,t)$ with *respect to x*

$$u_x = \frac{\partial}{\partial x}\{u(x,t)\} \ , \ u_{xx} = \frac{\partial^2}{\partial x^2}\{u(x,t)\}$$

(iii) Derivative of $u(x,t)$ with *respect to t and x*

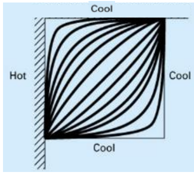
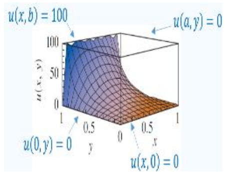
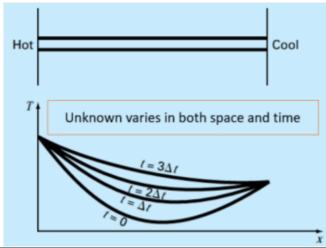
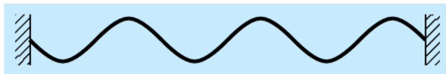
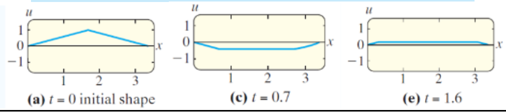
$$u_{tx}\ or\ u_{xt} = \frac{\partial^2}{\partial x \partial t}\{u(x,t)\}$$

Thus, we can rewrite and simplify the previous PDE using this notation:

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0 \quad \text{, where } A - F \text{ are constants.}$$

12.2 CATEGORIES OF 2ND ORDER LINEAR HOMOGENEOUS PDE

Based on the $B^2 - 4AC$, the PDE can be categorized into 3 types:

Category	Example	Application
Elliptic PDE $B^2 - 4AC < 0$	Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow u(x, y) = ?$ Characteristic: Steady state/ Time invariant	To find the stable temperature distribution of a heated/cooled 2D plate 
Parabolic PDE $B^2 - 4AC = 0$	Heat conduction equation / Heat equation $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \rightarrow u(x, t) = ?$ Characteristic: Time variant, non-oscillating	To find the temperature of a heated/cooled 1D rod that changes over time without oscillation 
Hyperbolic PDE $B^2 - 4AC > 0$	Wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \rightarrow u(x, t) = ?$ Characteristic: Time variant, oscillating	To find the vibration of a string that changes over time with oscillation 

Note: $\frac{\partial}{\partial t}$ = Time variant (change with time) or transient behavior

Time invariant means that the physical quantity will not change with time or steady state behavior

Description	**Elliptic PDE Equation**	**Strategy to solve**
One-dimensional Laplace equation	$\frac{\partial^2 u}{\partial x^2} = 0$ $u(x) = ?$	Integration, Solve PDE like ODE, Reduction of Order (Out of the scope – Appendix 12 for extra info)
Two-dimensional Laplace equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $u(x, y) = ?$	Separation of variables method (**Focus**)
Three-dimensional Laplace equation	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ $u(x, y, z) = ?$	Separation of variables method can be applied to 3D cases, however these 2 cases (Out of the scope)

Description	Parabolic PDE Equation	Strategy to Solve
One-dimensional heat equation	$\frac{1}{c^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ $u(x,t)=?$	Separation of variables method (**Focus**)
Two-dimensional heat equation	$\frac{1}{c^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ $u(x,y,t)=?$	Separation of variables method can be applied to 2D and 3D cases, however these 2 cases (out of the scope)
Three-dimensional heat equation	$\frac{1}{c^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ $u(x,y,z,t)=?$	

Description	Hyperbolic PDE Equation	Strategy to solve
One-dimensional wave equation	$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$ $u(x,t)=?$	Separation of variables method (**Focus**)
Two-dimensional wave equation	$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ $u(x,y,t)=?$	Separation of variables method can be applied to 2D and 3D cases, however these 2 cases (out of the scope)
Three-dimensional wave equation	$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$ $u(x,y,z,t)=?$	

Note that solving non-homogeneous PDE problem is out of scope in this study.

For example: The non-zero RHS function, $f(x,y)\ or\ f(x,t)\neq 0$

Description	Non-Homogeneous Elliptic PDE Equation
Two-dimensional Poisson equation with heat source/sink	$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$

Description	Non-Homogeneous Parabolic PDE Equation
One-dimensional heat equation with heating element	$\frac{\partial u}{\partial t}-c^2\frac{\partial^2 u}{\partial x^2}=f(x,t)$

Description	Non-Homogeneous Hyperbolic PDE Equation
One-dimensional wave equation with forcing function	$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=f(x,t)$

12.3 SEPARATION OF VARIABLE METHOD

For the 2nd order linear homogeneous PDE problem with constant coefficients:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

We assume that our solution to be:

$$u(x,y)=\underbrace{X(x)Y(y)}_{can\ be\ separated\ into\ x\ function\ and\ y\ function\ respectively}$$

Differentiate it,

$\frac{\partial u}{\partial x}=\frac{\partial X(x)}{\partial x}Y(y)=X'Y$; $\frac{\partial u}{\partial y}=X(x)\frac{\partial Y(y)}{\partial y}=XY'$	
$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 X(x)}{\partial x^2}Y(y)=X''Y$; $\frac{\partial^2 u}{\partial y^2}=X(x)\frac{\partial^2 Y(y)}{\partial y^2}=XY''$; $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial X(x)}{\partial x}\frac{\partial Y(y)}{\partial y}=X'Y'$	

$$AX''Y+BX'Y'+CXY''+DX'Y+EXY'+FXY=0$$

$$Y(AX''+DX'+FX)+Y'(BX'+EX)+Y''(CX)=0$$

If the PDE can be simplified to $C=0$; $B\ \&\ E=0$; or $A,D\ \&\ F=0$. The problem can be simplified to a separable differential equation. For example,

- If $C=0$ >> $Y(AX''+DX'+FX)+Y'(BX'+EX)=0$

Rearrange the equation, we get the separation of variable result:

$$\frac{Y'}{Y}=\frac{-(AX''+DX'+FX)}{(BX'+EX)}=-\lambda$$

By assuming it is equal to a separation constant of $-\lambda$, we success to convert it into 2 ODE equations.

ODE #1: $\frac{Y'}{Y}=-\lambda$	**ODE #2**: $\frac{-(AX''+DX'+FX)}{(BX'+EX)}=-\lambda$
$Y'+\lambda Y=0$	$(AX''+DX'+FX)=(\lambda BX'+\lambda EX)$ $(A)X''+(D-\lambda B)X'+(F-\lambda E)X=0$

Hint: Based on experience, separation constant of $-\lambda$ can solve the problem easier. In fact, let *separation constant of λ* can also solve the problem with same answer but longer procedure.

The separation constant, λ may be (i) zero, (ii) negative or (iii) positive. We can get three PDE solutions from these 3 cases.

Case #1 (λ=0)	**Case #2** ($\lambda=-\alpha^2$), $\alpha>0$	**Case #3** ($\lambda=+\alpha^2$), $\alpha>0$
$Y'=0$ $Y_1(y)=?$	$Y'-\alpha^2Y=0$ $Y_2(y)=?$	$Y'+\alpha^2Y=0$ $Y_3(y)=?$
$AX''+DX'+FX=0$ $X_1(x)=?$	$AX''+(D+\alpha^2B)X'+(F+\alpha^2E)X=0$ $X_2(x)=?$	$AX''+(D-\alpha^2B)X'+(F-\alpha^2E)X=0$ $X_3(x)=?$
$u_1=X_1(x)Y_1(y)$	$u_2=X_2(x)Y_2(y)$	$u_3=X_3(x)Y_3(y)$
Total PDE solution can be obtained by superposition principle: $u(x,y)=c_1u_1+c_2u_2+c_3u_3$		

Recall for the 2nd order linear homogeneous ODE:

$$aX'' + bX' + cX = 0$$

Assume solution, $X = e^{rx}$, Let r =root

Characteristic equation: $ar^2 + br + c = 0$

Root of the characteristic equation, $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (Note: you get 2 roots for 2nd order ODE)

For 2nd order ODE	If $r_1 \neq r_2$ for case of distinct roots or complex conjugate roots $X_1 = e^{r_1x}$; $X_2 = e^{r_2x}$ where X_1 & X_2 are linearly independent	Total solution can be obtained by superposition principle without treatment: $X_{total} = c_1e^{r_1x} + c_2e^{r_2x}$
	f $r_1 = r_2$ for case of repeated roots $X_1 = e^{r_1x}$; $X_2 = e^{r_2x} = e^{r_1x}$ where X_1 & X_2 are linearly dependent Treatment must be done by multiplying with the independent variable $X_1 = e^{r_1x}$; $X_{2,treat} = xe^{r_2x} = xe^{r_1x}$ where X_1 & $X_{2,treat}$ are linearly independent	Total solution can be obtained by superposition principle with treatment: $X_{total} = c_1e^{r_1x} + c_2xe^{r_2x}$

Note that the same method can be used to solve 1st order linear homogeneous ODE:

$$bX' + cX = 0$$

Assume solution, $X = e^{rx}$, Let r =root

Characteristic equation: $br + c = 0$

Root of the characteristic equation, $r = \frac{-c}{b}$ (Note: you get 1 root for 1st order ODE)

For 1st order ODE	$X_1 = e^{r_1x}$	Total solution can be obtained by superposition principle: $X_{total} = c_1e^{r_1x}$

Example: Solve the general solution of PDE below by using the separation of variable method

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

- ***Step 1:*** *Using separation of variable method:* Let $u(x,y) = X(x)Y(y)$

$$X''Y = 4XY'$$

- ***Step 2:*** *Obtain 2 ODE equations*

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

($Hint$: Calculation is easier with the coefficient = 1 for the numerator components)

$$Y' + \lambda Y = 0 \quad \text{--- (ODE \#1)}$$

$$X'' + 4\lambda X = 0 \quad \text{--- (ODE \#2)}$$

- ***Step 3:*** *3 cases of* λ

<u>3.1 Case #1 (λ=0)</u>

$Y' = 0$	$X'' = 0$
Let r =root Characteristic equation: $r = 0$ $\therefore Y(y) = c_1 e^{ry} = c_1$	Let r =root Characteristic equation: $r^2 = 0$, Repeated root case, $r_1 = r_2 = 0$ $\therefore X(x) = c_2 e^{r_1 x} + c_3 \underbrace{x}_{treament} e^{r_2 x}$ $X(x) = c_2 + c_3 x$
PDE solution in Case 1: $\therefore u_1 = X_1(x)Y_1(y) = (c_2 + c_3 x)(c_1) = A_1 x + B_1 \; where\, A_1, B_1 = constant$	

<u>3.2 Case #2 (λ= $-\alpha^2$), $\alpha > 0$</u>

$Y' - \alpha^2 Y = 0$	$X'' - 4\alpha^2 X = 0$
Let r =root *Characteristic equation:* $r - \alpha^2 = 0$ $r = \alpha^2$ $\therefore Y(y) = c_4 e^{\alpha^2 y}$	Let r =root *Characteristic equation:* $r^2 - 4\alpha^2 = 0$ $r = \pm\sqrt{4\alpha^2}$ Distinct root case: $r_1 = +2\alpha, r_2 = -2\alpha$ $\therefore X(x) = c_5 e^{2\alpha x} + c_6 e^{-2\alpha x}$
PDE solution in Case 2: $\therefore u_2 = X_2(x)Y_2(y) = (c_5 e^{2\alpha x} + c_6 e^{-2\alpha x})(c_4 e^{\alpha^2 y})$ $u_2 = e^{\alpha^2 y}(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}) \; where\, A_2, B_2 = constant$	

3.3 Case #3 ($\lambda = +\alpha^2$), $\alpha > 0$

$Y' + \alpha^2 Y = 0$	$X'' + 4\alpha^2 X = 0$
Let r =root *Characteristic equation:* $r + \alpha^2 = 0$ $r = -\alpha^2$ $\therefore Y(y) = c_7 e^{-\alpha^2 y}$	Let r =root *Characteristic equation:* $r^2 + 4\alpha^2 = 0$ $r = \pm\sqrt{-4\alpha^2}$ Complex conjugate root case: $r_1 = +2\alpha i,\ r_2 = -2\alpha i$ $\therefore X(x) = c_8 e^{2\alpha x i} + c_9 e^{-2\alpha x i}$
PDE solution in Case #3: $\therefore u_3 = X_3(x)Y_3(y) = \left(c_8 e^{2\alpha x i} + c_9 e^{-2\alpha x i}\right)\left(c_7 e^{-\alpha^2 y}\right)$ $u_3 = e^{-\alpha^2 y}\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right)$ *where* A_3, B_3 =*constant*	

- ***Step 4:*** *Using superposition principle to find the general PDE solution*

$$u(x,y) = \underbrace{A_1 x + B_1}_{Case\ 1\ solution} + \underbrace{e^{\alpha^2 y}\left(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}\right)}_{Case\ 2\ solution} + \underbrace{e^{-\alpha^2 y}\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right)}_{Case\ 3\ solution}$$

12.4 EXPRESSION OF PDE SOLUTION IN TERMS OF COS/SINE OR COSH/SINH

Previously in ODE chapter, we have learned that the exponential of complex conjugate roots can be expressed in terms of cos and sin via Euler formula.

$$\left(A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}\right) = \left(C_3 cos(2\alpha x) + D_3 sin(2\alpha x)\right)$$

Similarly, exponential of distinct real roots can be expressed in terms of $cosh$ and $sinh$ via Euler formula. These two expressions are useful to find the particular solution for the PDE later.

$$\left(A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}\right) = \left(C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)\right)$$

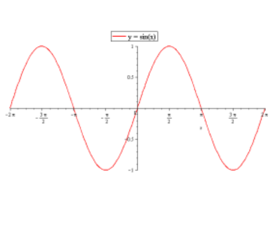
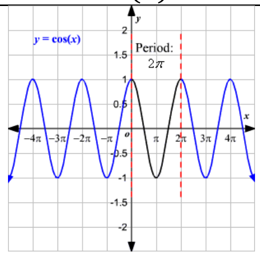
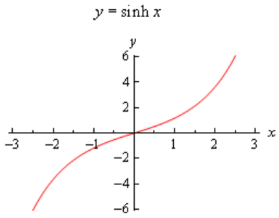
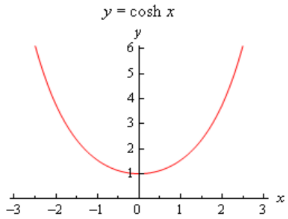
Derivation by using Euler Formula is given below:

$A_3 e^{2\alpha x i} + B_3 e^{-2\alpha x i}$	$A_2 e^{2\alpha x} + B_2 e^{-2\alpha x}$
Since $e^{i\theta} = cos\theta + isin\theta$, we get $\therefore A_3\left(cos(2\alpha x) + isin(2\alpha x)\right)$ $+B_3\left(cos(-2\alpha x) + isin(-2\alpha x)\right)$	Since $e^{\theta} = cosh\theta + sinh\theta$, we get $\therefore A_2\left(cosh(2\alpha x) + sinh(2\alpha x)\right)$ $+B_2\left(cosh(-2\alpha x) + sinh(-2\alpha x)\right)$
Since $cos(-\theta) = cos\theta, sin(-\theta) = -sin\theta$ $\therefore A_3\left(cos(2\alpha x) + isin(2\alpha x)\right)$ $+B_3\left(cos(2\alpha x) - isin(2\alpha x)\right)$	Since $cos\,h(-\theta) = cosh\theta, sinh(-\theta) = -sinh\theta$ $\therefore A_2\left(cosh(2\alpha x) + sinh(2\alpha x)\right)$ $+B_2\left(cosh(2\alpha x) - sinh(2\alpha x)\right)$
Rearrange, $\therefore cos(2\alpha x)(A_3 + B_3)$ $+sin(2\alpha x)(iA_3 - iB_3)$ $\therefore C_3 cos(2\alpha x) + D_3 sin(2\alpha x)$	Rearrange, $\therefore cosh(2\alpha x)(A_2 + B_2)$ $+sinh(2\alpha x)(A_2 - B_2)$ $\therefore C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)$
where $C_3 = A_3 + B_3; D_3 = iA_3 - iB_3$	where $C_2 = A_2 + B_2; D_2 = A_2 - B_2$

Thus, the previous PDE solution can be expressed in the cos/sine & cosh/sinh formats:

$$u(x,y) = \underbrace{A_1 x + B_1}_{Case\ 1\ solution} + \underbrace{e^{\alpha^2 y}\left(C_2 cosh(2\alpha x) + D_2 sinh(2\alpha x)\right)}_{Case\ 2\ solution} + \underbrace{e^{-\alpha^2 y}\left(C_3 cos(2\alpha x) + D_3 sin(2\alpha x)\right)}_{Case\ 3\ solution}$$

Important Characteristics of Sine, Cosine, Hyperbolic Sine & Hyperbolic Cosine:

Sine, $sin(x)$	Cosine, $cos(x)$	Hyperbolic Sine, $sinh(x)$	Hyperbolic Cosine, $cosh(x)$
			
sin(0)=0	*cos(0)=1*	*sinh(0)=0*	*cosh(0)=1*
Odd function *sin(-x)= -sin(x)*	*Even function* *cos(-x)=cos(x)*	*Odd function* *sinh(-x)= -sinh(x)*	*Even function* *cosh(-x)=cosh(x)*
$\frac{d}{dx}sin(x) = cos(x)$	$\frac{d}{dx}cos(x) = -sin(x)$	$\frac{d}{dx}sinh(x) = cosh(x)$	$\frac{d}{dx}cosh(x) = sinh(x)$
$sin(n\pi) = 0$ where $n = integer$	$cos\left((2n-1)\frac{\pi}{2}\right) = 0$ where $n = integer$	$sinh(0) = 0$ *only when* $x = 0$ $sinh(x) > 0$ *for* $x > 0$	$cosh(x) \neq 0$ *for any* x $cosh(x) > 0$ *for any* x

12.5 INITIAL/ BOUNDARY CONDITION OF PDE PROBLEM

Previously, we solve the following PDE: $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$ and obtain the general solution with a lot of unknowns $(A_1, B_1, A_2, B_2, A_3, B_3)$. By using initial or/and boundary conditions of the problem, we can continue to solve those unknowns and obtain the particular solution of the PDE.

Thus, it is important to formulate the initial/ boundary condition from a given problem. The three conditions that are found to occur most regularly are

Cauchy conditions

u and $\frac{\partial u}{\partial n}$ given on C

Dirichlet conditions

u given on C

Neumann conditions

$\frac{\partial u}{\partial n}$ given on C

Note: A boundary C is said to be closed if conditions are specified on the whole of it, or open if conditions are only specified on part of it. Naming of the type of conditions is out of scope, it is sufficient as long as student is able to formulate the equations for the initial/ boundary conditions.

Example of **formulating the initial/ boundary condition** from Elliptic PDE, Parabolic PDE, and Hyperbolic PDE are given below:

i. **Elliptic PDE**: Set up the boundary value problem for the steady-state temperature $u(x, y)$ for a thin rectangular plate coincides with the region defined by $0 \le x \le 4$, $0 \le y \le 2$. The left end and the bottom of the plate are insulated. The top of the plate is held at temperature zero, and the right end of the plate is held at temperature $f(y)$. The PDE that governs the problem is given:

2D Laplace Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Solution: Stable temperature distribution of 2D plate, $u(x, y)$

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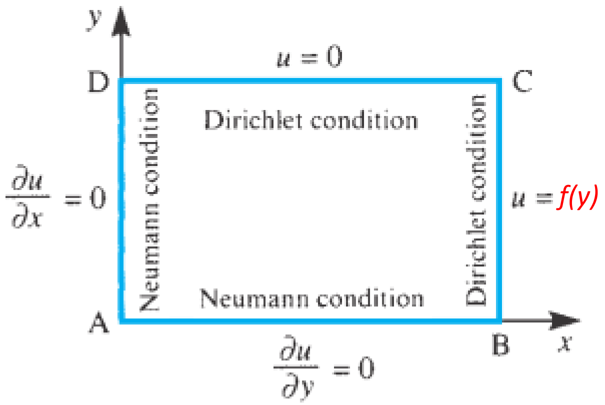
The corresponding region and boundary conditions in the (x,y) plane for a steady state heated rectangular plate.



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<u>Dirchlet condition for the top and right end:</u>

$u(x, 2) = 0\,, \quad 0 < x < 4$

$u(4,\ y) = f(y)\,, \quad 0 < y < 2$

<u>Neumann condition for the bottom and left end:</u>

$\frac{\partial u(x,0)}{\partial y} = \frac{\partial u}{\partial y}\Big|_{y=0} = 0\,,\ 0 < x < 4$

$\frac{\partial u(0,y)}{\partial x} = \frac{\partial u}{\partial x}\Big|_{x=0} = 0,\ \ 0 < y < 2$

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Note: The heat flux can't flow in the x-direction, $q_x = 0$ if there is insulation on the left end. Since $q_x = -k \underbrace{\frac{\partial u}{\partial x}}_{Temperature\ gradient} = 0$, thus $\frac{\partial u}{\partial x}\Big|_{x=0} = 0$ indicates insulation on left end which blocks the heat flux to flow in the x-direction.

ii. **Parabolic PDE:** A rod of length *L* coincides with the interval [0, *L*] on the *x*-axis. <u>Set up the boundary value problem</u> for the temperature *u*(*x*, *t*) when the left end is held at temperature zero, and the right end is insulated. The initial temperature is *f*(*x*) throughout. The PDE that governs the problem is given: *1D Heat Equation* $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where *c* = constant. Formulate the initial/ boundary condition.

Solution: Temperature of the 1D bar that changes over time, $u(x, t)$

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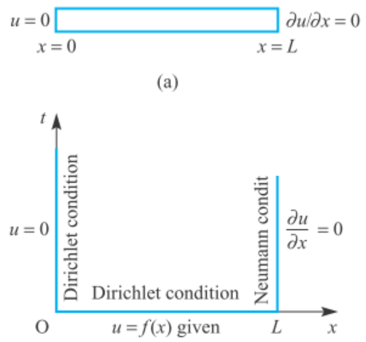
The corresponding region and boundary conditions in the (x,t) plane for a heated/cooled bar



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<u>Dirchlet condition for the left end:</u>

$u(0,\ t) = 0\,, t > 0$

<u>Dirchlet condition for the initial temperature:</u>

$\left.\begin{matrix} u(x,\ 0) = f(x) \\ \underbrace{\frac{\partial u(x,t)}{\partial x} = \frac{\partial u}{\partial x}\Big|_x \neq 0}_{bar\ is\ not\ insulated} \end{matrix}\right\} \ 0 < x < L$

<u>Neumann condition for the right end:</u>

$u_x(L, t) = \frac{\partial u(L, t)}{\partial x} = \frac{\partial u}{\partial x}\Big|_{x=L} = 0\,, t > 0$

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iii. **Hyperbolic PDE:** *A string of length L coincides with interval [0, L] on the x-axis. Set up the boundary value problem for the displacement u(x, t) when the ends are secured to the x-axis. The string is released from rest from the initial displacement x(L – x).* The PDE that governs the problem is given: *1D Wave Equation* $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where c = constant. Formulate the initial/ boundary condition.

Solution: Vibration of the 1D string over time, $u(x,t)$

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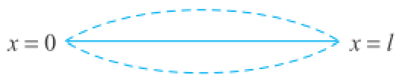
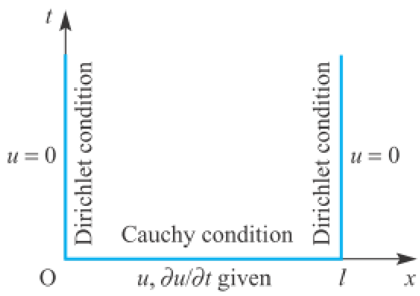
The corresponding region and boundary conditions in the (x,t) plane for a vibrating string





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<u>Dirchlet condition for the fixed end:</u>

$$\left.\begin{aligned} u(0,\ t) &= 0 \\ u(L,\ t) &= 0 \end{aligned}\right\} t > 0$$

<u>Cauchy condition for initial displacement/velocity:</u>

$$\left.\begin{aligned} u(x,\ 0) &= x(L-x) \\ \frac{\partial u(x,0)}{\partial t} = \left.\frac{\partial u}{\partial t}\right|_{t=0} &= 0 \end{aligned}\right\} \ 0 < x < L$$

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