SOLVING PARTICULAR SOLUTION OF HEAT EQUATION & WAVE EQUATION

WEEK 14: SOLVING PARTICULAR SOLUTION OF HEAT EQUATION & WAVE EQUATION

14.1 STRATEGY TO SOLVE HOMOGENEOUS PDE PROBLEM VIA SEPARABLE OF VARIABLE

Previously we have learned how to apply separation of variable method to solve the **Laplace equation**, then we formed the boundary conditions of the problem and apply it together with the Fourier series expansion to obtain the particular PDE solution. Same strategy is used to solve the **heat equation** and **wave equation** in this chapter, as summarized below:

Let $u =$ dependent variable, $x, t =$ independent variables

Step 1: $u(x,t) = X(x)T(t)$

Step 2: Obtains 2 ODE equations using separation constant, $-\lambda$. Let the coefficient of numerator to be 1 for easier calculation.

Step 3: Consider 3 cases: $\lambda=0$; $\lambda=-\alpha^2$; $\lambda=\alpha^2$, where $\alpha > 0$

Then, we can obtain all possible solutions, u_1, u_2, & u_3 respectively for each case.

Step 4.1: If initial/ boundary conditions can't be formed/ obtained,

General PDE solution via superposition principle, $u(x,y) = c_1u_1 + c_2u_2 + c_3u_3$,

where c_1, c_2, c_3 are unknowns.

Step 4.2: If initial/ boundary conditions can be formed/ obtained,

Then, we proceed to apply the homogeneous BC to solve the particular solution, u_1, u_2, & u_3 for each case. Then, eigenvalue and eigenfunction can be identified for case with solution and they can be combined to form the total solution.

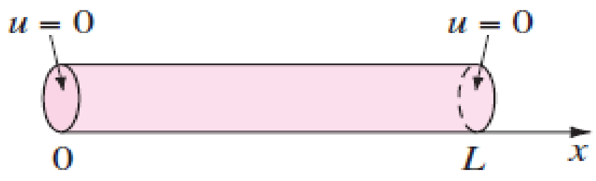
Step 5: Continue to apply the remaining initial/ boundary conditions & Fourier series expansion to solve the remaining unknown.

Particular PDE solution via superposition principle, $u(x,y) = u_1 + u_2 + u_3$,

where c_1, c_2, c_3 are found.

14.2 SOLVING PARTICULAR SOLUTION OF PARABOLIC PDE (HEAT EQUATION)

Consider a thin rod of length L with an initial temperature f(x) throughout and whose ends are held at temperature zero for all time t > 0. Given these initial/boundary conditions, find the <u>change of the temperature over the time and x location</u>, i.e. $u(x,t)$.



1D rod with boundary conditions on both ends and initial temperature of the bar.

- **Governing equation for the 1D heat equation**

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

- **Boundary condition 1 & 2**: $u(0,\ t) = 0\ ,u(L,\ t) = 0$ for $t > 0$

- **Initial condition** : $u(x,\ 0) = f(x)$ for $0 < x < L$

Solution:

Step 1: *Using separation of variable method:* Let $u(x,t) = X(x)T(t)$

$$kX''T = XT'$$

Step 2: *Obtain 2 ODE equations*

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$$

$T' + k\lambda T = 0$ --- (ODE #1)

$X'' + \lambda X = 0$ --- (ODE #2)

Case	ODE #1	ODE #2	$u(x,y) = X(x)T(t)$
Case #1: (λ=0)	$T' = 0$ Let r =root Characteristic equation: $r = 0$ $T(t) = c_1e^{0t}$ $\therefore T(t) = c_1$	$X'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated root: $r_1 = 0,\ r_2 = 0$ $X(x) = c_2e^{0x} + c_3xe^{0x}$ $\therefore X(x) = c_2 + c_3x$	$\therefore u_1 = X_1(x)T_1(t)$ $= (c_1)(c_2 + c_3x)$ $= A_1x + B_1$

Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$	$T' - (\alpha^2 k)T = 0$ Let r =root Characteristic equation: $r - \alpha^2 k = 0$ $r = \alpha^2 k$ $\therefore T(t) = c_4 e^{\alpha^2 kt}$	$X'' - \alpha^2 X = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 = 0$ $r = \pm\sqrt{\alpha^2} = \pm\alpha$ Distinct roots: $r_1 = \alpha, \quad r_2 = -\alpha$ $\therefore X(x) = c_5 cosh(\alpha x) + c_6 sinh(\alpha x)$	$\therefore u_2 = X_2(x)T_2(t)$ $= c_4 e^{\alpha^2 kt}\,(c_5 cosh(\alpha x) + c_6 sinh(\alpha x))$ $= e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))$
Case #3: ($\lambda = +\alpha^2$) $\alpha > 0$	$T' + \alpha^2 kT = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 k = 0$ $r = -\alpha^2 k$ $\therefore T(t) = c_7 e^{-\alpha^2 kt}$	$X'' + (\alpha^2)X = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 = 0$ $r = \pm\sqrt{-\alpha^2} = \pm\alpha i$ Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$ $\therefore X(x) = c_8 cos(\alpha x) + c_9 sin(\alpha x)$	$\therefore u_3 = X_3(x)T_3(t)$ $= \left(c_7 e^{-\alpha^2 kt}\right)\left(c_8 cos(\alpha x) + c_9 sin(\alpha x)\right)$ $= e^{-\alpha^2 kt}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)$

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,t) = \underbrace{A_1 x + B_1}_{Solution\ of\ Case\ 1} + \underbrace{e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))}_{Solution\ of\ Case\ 2} + \underbrace{e^{-\alpha^2 kt}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)}_{Solution\ of\ Case\ 3}$$

where there are 6 unknown coefficients ($A_1 - B_3$). Next, we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions.

Boundary condition (BC) #1: $u(0,\ t) = 0$, BC #2: $u(L,\ t) = 0$

Case	Applying BC #1 & BC #2
Case #1: (λ=0)	$u_1 = X_1(x)T_1(t)$ $= A_1 x + B_1$ Applying BC #1: $u_1(0,t) = A_1(0) + B_1 = 0$ Thus, $B_1 = 0$ $\rightarrow u_1 = A_1 x$ Applying BC #2, we get $u_1(L,t) = A_1 L = 0$ Since $L \neq 0$, $A_1 = 0$ $\therefore u_1(x,t) = 0$ (No solution)

Case #2: (λ= $-\alpha^2$) $\alpha > 0$	$u_2 = X_2(x)T_2(t)$ $= e^{\alpha^2 kt}\,(A_2 cosh(\alpha x) + B_2 sinh(\alpha x))$ Applying BC #1: $u_2(0,t) = e^{-\alpha^2 kt}\left(A_2\right) = 0$ Comparing coefficient $e^{-\alpha^2 kt}$: thus $A_2 = 0$ (Note: $e^{-\alpha^2 kt} \neq 0$ as the temperature changes over time, else no solution) $\rightarrow u_2(x,t) = B_2 sinh(\alpha x)$ Applying BC #2: $u_2(L,t) = e^{-\alpha^2 kt}\left(B_2 sinh(\alpha L)\right) = 0$ Comparing coefficient $e^{-\alpha^2 kt}$: $\left(B_2 sinh(\alpha L)\right) = 0$ Since $sinh(\alpha L)$ will not be zero for $\alpha > 0$, thus $B_2 = 0$ Hint: $\alpha L > 0$ $\therefore u_2(x,t) = 0$ (No solution)
Case #3: (λ= $+\alpha^2$) $\alpha > 0$	$\therefore u_3 = X_3(x)T_3(t)$ $= e^{-\alpha^2 kt}\left(A_3 cos(\alpha x) + B_3 sin(\alpha x)\right)$ Applying BC #1: $u_3(0,t) = e^{-\alpha^2 kt}\left(A_3\right) = 0$ Comparing coefficient $e^{-\alpha^2 kt}$: $A_3 = 0$ $\rightarrow u_3 = e^{-\alpha^2 kt}\left(B_3 sin(\alpha x)\right)$ Applying BC #2: $u_3(L,t) = e^{-\alpha^2 kt}\left(B_3 sin(\alpha L)\right) = 0$ Comparing coefficient $e^{-\alpha^2 kt}$: $\left(B_3 sin(\alpha L)\right) = 0$ Since $B_3 \neq 0$ *when* $sin(\alpha L) = 0$ for $\alpha L = n\pi$, where $\alpha = \frac{n\pi}{L}$, $n = 1,2,3 \ldots$ There are infinite solutions in Case #3: $u_{3,n} = e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$ *where n =1,2,3,...*

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

Case	PDE solution	Eigenvalue and eigenfunction of PDE
Case #1: (λ=0)	$u_1(x,t) = 0$	No solution hence no eigenvalue and no eigenfunction
Case #2: (λ= $-\alpha^2$) $\alpha > 0$	$u_2(x,t) = 0$	No solution hence no eigenvalue and no eigenfunction
Case #3: (λ= $+\alpha^2$) $\alpha > 0$	$u_{3,n} = e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$	*Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{L}\right)^2$ *Eigenfunction* $u_{3,n}$ $= e^{-\left(\frac{n\pi}{L}\right)^2 kt}\left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$

Step 4: Superposition Principle to find $u_{total}(x,t) = X_1T_1 + X_2T_2 + X_3T_3$

$$u_{total}(x,t) = \underbrace{\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left(B_{3,n} sin(\frac{n\pi}{L}x)\right)}_{solution\ from\ Case\ 3}$$

where there are 1 unknown remaining (i.e. $B_{3,n}$).

Step 5: Continue to apply the remaining BC & Fourier series expansion.

BC #3: $u(x,\ 0) = f(x)$ for $0 < x < L$

$u_{total}(x,0) = \sum_{n=1}^{\infty} \left(B_{3,n} sin(\frac{n\pi}{L}x)\right) = f(x)$

Recall Half-range Fourier Sine Series Expansion:

$f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$
where $b_n = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x\, dx$

Precaution: L in the formula indicates the half period, i.e. $L = \frac{p}{2} = \frac{\pi}{\omega}$. Do not mix it with the length of the 1D bar, which is using the same symbol, L as well.

Note that for (i) Half-range expansion: Finite interval, $\tau = half\ period, L$
(ii) Full-range expansion: Finite interval, $\tau = full\ period, 2L$

We notice $B_{3,n} = b_n = \frac{2}{L}\int_0^{\tau} f(x) \sin n\omega x\, dx$,

where $\omega = \frac{\pi}{L}$ &

From $0 < x < L, \tau = \ length, L$. For half-range expansion, $\tau = \ half\ period, L$. Thus, in this case it happens to have $\tau = half\ period, L = length, L$ in this special case.

Precaution: Note that it would be different for full-range expansion case.

$\rightarrow B_{3,n} = \frac{2}{L}\int_0^{L} f(x) \sin n\frac{\pi}{L}x\, dx$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left(B_{3,n} sin(\frac{n\pi}{L}x)\right)$$

$$u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \left(\frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx\, sin(\frac{n\pi}{L}x) \right)$$

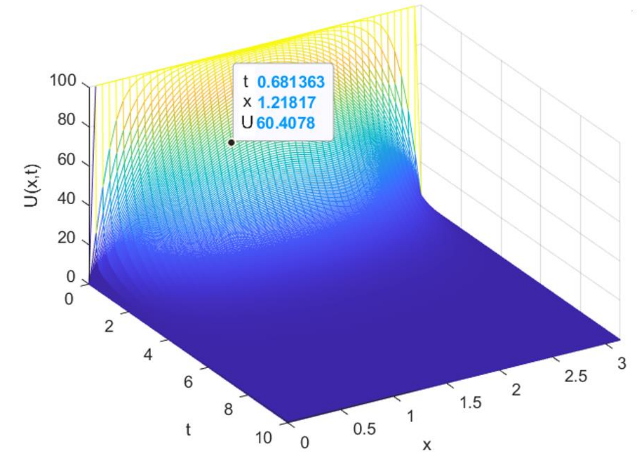
Example: Let the $f(x) = 100$, dimension, $length, L = \pi$, PDE coefficient, $k = 1$ for the previous problem.

$$B_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$$
$$= \frac{2}{\pi}\int_0^{\pi} 100\, sin\, nx\, dx$$
$$= \frac{200}{\pi}\left[\frac{-cosnx}{n}\right]_0^{\pi}$$
$$= \frac{200}{\pi}\left(\frac{-cosn\pi}{n} - \frac{-1}{n}\right)$$
$$= \frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{\pi}\right)^2 (1)t} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(\frac{n\pi}{\pi}x) \right)$$
$$= \sum_{n=1}^{\infty} e^{-n^2 t} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(nx) \right)$$

We can use the PDE solution to estimate the temperature distribution at any point on the cooled rod. Example: The temperature results at 50×500 points of the (x,t) locations for a duration of 10s have been plotted below:

3D plot of $u_{total}(x,t)$ wrt x- & t- axes	2D contour plot

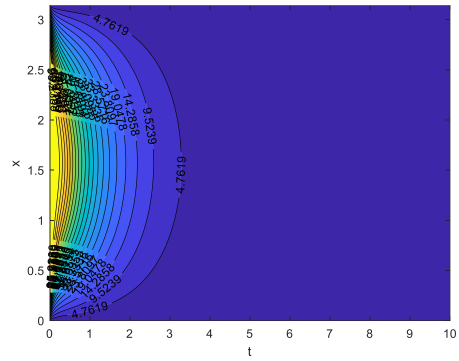


-Due to boundary conditions on both sides of the 1D rods and the initial temperature of the rod, the temperature of the rod changes over time, and it is converging to the BC temperature, i.e. $0\ ^oC$.

For example, $u_{total}(1.218, 0.681) \approx 60\ ^oC$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grids can be included but computational time will be increased.

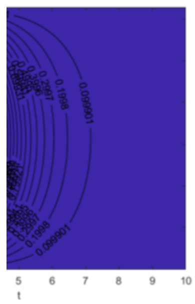
Try to verify the answer:

$$u_{total}(1.218, 0.681) \approx \sum_{n=1}^{20} e^{-n^2(0.681)} \left(\frac{200}{\pi}\left(\frac{1-(-1)^n}{n}\right) sin(n(1.218))\right)$$



-Top view of the 3D plot with the contour, i.e. line with same magnitude.

Try to find the location of the plate that drops to $4.7619\ ^oC$ last.
Hint: max of u_{total} at fixed time ; sin characteristic.



By increasing the contours, we can observe that the temperatures of the whole bar takes around 7s, in order to drop to less than $0.1\ ^oC$ due to BC on both ends.

Relationship between Laplace equation and heat equation:

In the heat equation example:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

We observe that the temperature results become stable/ no change after some durations. This means that $\frac{\partial u}{\partial t} = 0$ for $t \to \infty$ (i.e. change of temperature, u over time is zero for sufficient large duration, t).

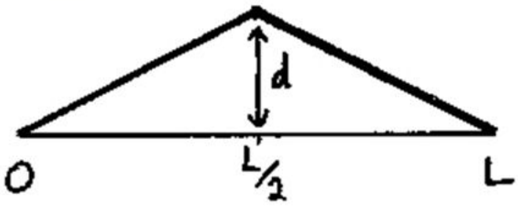
Depending on our application, we will go for

(i) Solving heat equation, $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ if we are interested in finding out the change of the temperature, u over time.
Note: The solution $u(x,t)$ contains the transient solution at beginning and steady state solution when $t \to \infty$.

(ii) Solving Laplace equation, $k\frac{\partial^2 u}{\partial x^2} = 0$ by let $\frac{\partial u}{\partial t} = 0$ only if we are interested in finding out the stable temperature without changes over time.
Note: The solution $u(x)$ contains the steady state solution only.

14.3 SOLVING PARTICULAR SOLUTION OF HYPERBOLIC PDE (WAVE EQUATION)

Consider a string of length L, stretched taut between 2 points on x-axis (e.g. x=0 and x=L)., find the change of vertical displacement with respect to time and x location, i.e. $u(x,t)$.



Transverse vibration u(x, t) in rod of length L

The string is fixed at both ends like guitar string.

- **Governing equation for the 1D wave equation**

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

- **Boundary condition #1 & #2**: $u(0,\, t) = 0\, ,u(L,\, t) = 0$ for $t > 0$

- **Initial condition #1 & #2** : $u(x,\, 0) = f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ for $0 < x < L$

  Note: For the string's vibration, $u(x,\, 0) =$ initial displacement, while $u_t(x,\, 0) =$ initial velocity.

Solution:

Step 1: *Using separation of variable method:* Let $u(x,t) = X(x)T(t)$

$$a^2 X''T = XT''$$

Step 2: *Obtain 2 ODE equations*

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$T'' + a^2\lambda T = 0$ --- (ODE #1)

$X'' + \lambda X = 0$ --- (ODE #2)

Case	ODE #1	ODE #2	$u(x,y) = X(x)T(t)$
Case #1: (λ=0)	$T'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated root:	$X'' = 0$ Let r =root Characteristic equation: $r^2 = 0$ Repeated root:	$\therefore u_1 = X_1(x)T_1(t) = (c_1 + c_2 t)(c_3 + c_4 x)$

	$r_1 = 0,\ r_2 = 0$ $T(t) = c_1 e^{0t} + c_2 t e^{0t}$ $\therefore T(t) = c_1 + c_2 t$	$r_1 = 0,\ r_2 = 0$ $X(x) = c_3 e^{0x} + c_4 x e^{0x}$ $\therefore X(x) = c_3 + c_4 x$	
Case #2: (λ= $-\alpha^2$) $\alpha > 0$	$T'' - (\alpha^2 a^2)T = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 a^2 = 0$ Distinct roots: $r_1 = \sqrt{\alpha^2 a^2} = \alpha a,$ $r_2 = -\sqrt{\alpha^2 a^2} = \alpha a$ $\therefore T(t) = c_5 cosh(\alpha at) + c_6 sinh(\alpha at)$	$X'' - \alpha^2 X = 0$ Let r =root Characteristic equation: $r^2 - \alpha^2 = 0$ $r = \pm\sqrt{\alpha^2} = \pm\alpha$ Distinct roots: $r_1 = \alpha, \quad r_2 = -\alpha$ $\therefore X(x) = c_7 cosh(\alpha x) + c_8 sinh(\alpha x)$	$\therefore u_2 = X_2(x)T_2(t)$ $= \left(c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at)\ \right).$ $(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$
Case #3: (λ= $+\alpha^2$) $\alpha > 0$	$T'' + \alpha^2 a^2 T = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 a^2 = 0$ Complex conjugate roots: $r_1 = \alpha a i,\ r_2 = -\alpha a i$ $\therefore T(t) = c_9 cos(\alpha at) + c_{10} sin(\alpha at)$	$X'' + (\alpha^2)X = 0$ Let r =root Characteristic equation: $r^2 + \alpha^2 = 0$ $r = \pm\sqrt{-\alpha^2} = \pm\alpha i$ Complex conjugate roots: $r_1 = \alpha i,\ r_2 = -\alpha i$ $\therefore X(x) = c_{11} cos(\alpha x) + c_{12} sin(\alpha x)$	$\therefore u_3 = X_3(x)T_3(t)$ $= \left(c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at)\ \right).$ $\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\ \right)$

In fact, we can find the general PDE solution to the problem by using superposition principle:

$$u(x,t) = \underbrace{\left(c_1 + c_2 t\ \right)\left(c_3 + c_4 x\ \right)}_{Solution\ of\ Case\ 1} + \underbrace{\left(c_5\, cosh(\alpha at) + c_6\, sinh(\alpha at)\right)\left(c_7 cosh(\alpha x) + c_8 sinh(\alpha x)\right)}_{Solution\ of\ Case\ 2}$$

$$+ \underbrace{\left(c_9\, cos(\alpha at) + c_{10}\, sin(\alpha at)\right)\left(c_{11} cos(\alpha x) + c_{12} sin(\alpha x)\ \right)}_{Solution\ of\ Case\ 3}$$

where there are 12 unknown coefficients ($c_1 - c_{12}$). Next, we will continue to solve those unknowns by applying the initial/ boundary conditions.

To apply the following boundary conditions.

Boundary condition (BC) #1: $u(0,\ t) = 0$, BC #2: $u(L,\ t) = 0$

Case	Applying BC #1 & BC #2
Case #1: (λ=0)	$u_1 = X_1(x)T_1(t)$ $= \left(c_1 + c_2 t\ \right)\left(c_3 + c_4 x\ \right)$ Applying BC #1: $u_1(0,t) = \left(c_1 + c_2 t\ \right)(c_3) = 0$ Note: vibration is changing wrt time, thus $T(t)$ ≠0 for non-trivial solution. Since $\left(c_1 + c_2 t\ \right) \neq 0$, thus $c_3 = 0$ $\rightarrow u_1 = (c_1 + c_2 t\ )(c_4 x\ )$

	Applying BC #2, we get $u_1(L,t) = (c_1 + c_2 t)(c_4 L) = 0$ Since $(c_1 + c_2 t) \neq 0$, $L \neq 0$, thus $c_4 = 0$ $\therefore u_1(x,t) = 0$ (No solution)
Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$	$u_2 = X_2(x)T_2(t)$ $= (c_5\, cosh(\alpha a t) + c_6\, sinh(\alpha a t))(c_7 cosh(\alpha x) + c_8 sinh(\alpha x))$ Applying BC #1: $u_2(0,t) = (c_5 cosh(\alpha a t) + c_6 sinh(\alpha a t))(c_7) = 0$ Since $(c_5 cosh(\alpha a t) + c_6 sinh(\alpha a t)) \neq 0$, thus $c_7 = 0$ $\rightarrow u_2(x,t) = (c_5\, cosh(\alpha a t) + c_6\, sinh(\alpha a t))(c_8 sinh(\alpha x))$ Applying BC #2: $u_2(L,t) = (c_5\, cosh(\alpha a t) + c_6\, sinh(\alpha a t))(c_8 sinh(\alpha L)) = 0$ Since $(c_5\, cosh(\alpha a t) + c_6\, sinh(\alpha a t)) \neq 0, sinh(\alpha L) \neq 0$ for $\alpha L > 0$, thus $c_8 = 0$ $\therefore u_2(x,t) = 0$ (No solution)
Case #3: ($\lambda = +\alpha^2$) $\alpha > 0$	$\therefore u_3 = X_3(x)T_3(t)$ $= (c_9\, cos(\alpha a t) + c_{10}\, sin(\alpha a t))(c_{11} cos(\alpha x) + c_{12} sin(\alpha x))$ Applying BC #1: $u_3(0,t) = (c_9\, cos(\alpha a t) + c_{10}\, sin(\alpha a t))(c_{11}) = 0$ Since $(c_9\, cos(\alpha a t) + c_{10}\, sin(\alpha a t)) \neq 0$, thus $c_{11} = 0$ $\rightarrow u_3 = (c_9\, cos(\alpha a t) + c_{10}\, sin(\alpha a t))(c_{12} sin(\alpha x))$ Applying BC #2: $u_3(L,t) = (c_9 cos(\alpha a t) + c_{10} sin(\alpha a t))(c_{12} sin(\alpha L)) = 0$ Since $(c_9\, cos(\alpha a t) + c_{10}\, sin(\alpha a t)) \neq 0$ and $c_{12} \neq 0\ when\ sin(\alpha L) = 0$ for $\alpha L = n\pi$, where $\alpha = \frac{n\pi}{L}$, $n = 1,2,3$ ... There are infinite solutions in Case #3: $u_{3,n} = \left(c_{9,n}\, cos\left(\frac{n\pi a}{L} t\right) + c_{10,n}\, sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$ *,where n =1,2,3,…*

In summary, the eigenvalue and eigenfunction of the PDE for each case are listed below:

Case	PDE solution	Eigenvalue and eigenfunction of PDE
Case #1: (λ=0)	$u_1(x,t) = 0$	No solution hence no eigenvalue and no eigenfunction
Case #2: ($\lambda = -\alpha^2$) $\alpha > 0$	$u_2(x,t) = 0$	No solution hence no eigenvalue and no eigenfunction
Case #3: ($\lambda = +\alpha^2$) $\alpha > 0$	$u_{3,n} = \left(c_{9,n}\, cos\left(\frac{n\pi a}{L} t\right) + c_{10,n}\, sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$	*Eigenvalue,* $\lambda_n = +{\alpha_n}^2 = \left(\frac{n\pi}{L}\right)^2$ *Eigenfunction* $u_{3,n} = \left(c_{9,n}\, cos\left(\frac{n\pi a}{L} t\right) + c_{10,n}\, sin\left(\frac{n\pi a}{L} t\right)\right)\left(c_{12,n} sin(\frac{n\pi}{L} x)\right)$

Step 4: Superposition Principle to find $u_{total}(x,t) = X_1T_1 + X_2T_2 + X_3T_3$

$$u_{total}(x,t) = \underbrace{\sum_{n=1}^{\infty}\left(c_{9,n}\cos\left(\frac{n\pi a}{L}t\right) + c_{10,n}\sin\left(\frac{n\pi a}{L}t\right)\right)\left(c_{12,n}\sin(\frac{n\pi}{L}x)\right)}_{solution\ from\ Case\ 3}$$

where there are 3 remaining unknowns (i.e. $c_{9,n}, c_{10,n}, \&\ c_{12,n}$).

By expanding it, we can reduce the unknowns into 2 (i.e. $A_{3,\text{n}}, B_{3,n}$), as shown in displacement solution.

$$u_{total}(x,y) = \sum_{n=1}^{\infty} A_{3,\text{n}}\cos\left(\frac{n\pi a}{L}t\right)\left(\sin(\frac{n\pi}{L}x)\right) + \left(B_{3,n}\sin(\frac{n\pi a}{L}t)\left(\sin(\frac{n\pi}{L}x)\right)\right)$$

Differentiate the displacement solution wrt t, then we obtain the velocity solution.

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-A_{3,n}\frac{n\pi a}{L}\sin(\frac{n\pi a}{L}t)\left(\sin(\frac{n\pi}{L}x)\right)\right) + \sum_{n=1}^{\infty}\left(B_{3,n}\frac{n\pi a}{L}\cos(\frac{n\pi a}{L}t)\left(\sin(\frac{n\pi}{L}x)\right)\right)$$

Step 5: Continue to apply the remaining IC & Fourier series expansion.

IC #1: $u(x,\ 0) = f(x)$ for $0 < x < L$

$u_{total}(x,0) = \sum_{n=1}^{\infty}\left(A_{3,n}\left(\sin(\frac{n\pi}{L}x)\right)\right) = f(x)$

Recall Half-range Fourier Sine Series Expansion:

> $f(x) = \sum_{n=1}^{\infty}(b_n \sin n\omega x)$
> where $b_n = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx$
>
> Precaution: L in the formula indicates the half period, i.e. $L = \frac{p}{2} = \frac{\pi}{\omega}$. Do not mix it with the length of the 1D string, which is using the same symbol, L as well.
>
> Note that for (i) Half-range expansion: Finite interval, $\tau = half\ period, L$
> (ii) Full-range expansion: Finite interval, $\tau = full\ period, 2L$

We notice $A_{3,n} = b_n = \frac{2}{L}\int_0^{\tau} f(x)\sin n\omega x\, dx$,

where $\omega = \frac{\pi}{L}$ &

From $0 < x < L, \tau =\ length, L$. For half-range expansion, $\tau =\ half\ period, L$. Thus, in this case it happens to have finite interval, $\tau = half\ period, L = length, L$ in this special case.

Precaution: Note that it would be different for full-range expansion case.

$\rightarrow A_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$

Step 5: Continue to apply the remaining IC & Fourier series expansion.

IC #2: $u_t(x,\ 0) = g(x)$ for $0 < x < L$

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty}\left(B_{3,n}\frac{n\pi a}{L}\left(sin(\frac{n\pi}{L}x)\right)\right) = g(x)$

Recall Half-range Fourier Sine Series Expansion:

$g(x) = \sum_{n=1}^{\infty}(b_n\, sin\, n\omega x)$ *where* $b_n = \frac{2}{L}\int_0^{\tau} g(x)\, sin\, n\omega x\, dx$

We notice $B_{3,n}\frac{n\pi a}{L} = b_n = \frac{2}{L}\int_0^{\tau} f(x)\, sin\, n\omega x\, dx$,

where $\omega = \frac{\pi}{L}$;

From $0 < x < L, \tau = \ length, L$. For half-range expansion, $\tau = \ half\ period, L$. Thus, in this case it happens to have finite interval, $\tau = half\ period, L = length, L$ in this special case.

$\rightarrow B_{3,n}\frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$

$\rightarrow B_{3,n} = \frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$

Thus, we have solved all the unknowns and obtain the particular PDE solution:

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,n}\cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(B_{3,n}sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$

$$u_{total}(x,t) = \sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx\, cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(\frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx\, sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$

Example: Let the initial displacement, $f(x) = x(L-x)$, initial velocity, $g(x) = 0$, dimension, $length, L = 1$, PDE coefficient, $a = 1$ for the previous problem.

$A_{3,n} = \frac{2}{L}\int_0^L f(x)\, sin\, n\frac{\pi}{L}x\, dx$	$B_{3,n} = \frac{2}{n\pi a}\int_0^L g(x)\, sin\, n\frac{\pi}{L}x\, dx$

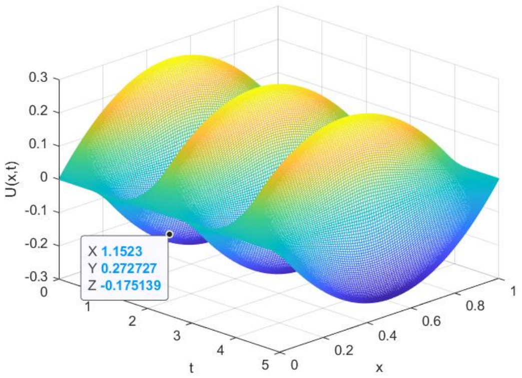
$= \frac{2}{1}\int_0^1 x(1-x)\, sin\, n\frac{\pi}{1}x\, dx$ $= 2\left[\int_0^1 x\, sin\, n\pi x\, dx - \int_0^1 x^2\, sin\, n\pi x\, dx\right]$ $= 2\left[\frac{\sin n\pi - n\pi cos n\pi}{n^2\pi^2} - \frac{2n\pi sin n\pi + (2-n^2\pi^2)cos n\pi - 2}{n^3\pi^3}\right]$ $= \left[-\frac{2n\pi sin n\pi + 4cos n\pi - 4}{n^3\pi^3}\right]$	$= \frac{2}{n\pi(1)}\int_0^1 (0)\, sin\, n\frac{\pi}{L}x\, dx$ $= 0$

$$\therefore u_{total}(x,t) = \sum_{n=1}^{\infty} A_{3,\mathrm{n}}\, cos\left(\frac{n\pi a}{L}t\right)\left(sin(\frac{n\pi}{L}x)\right) + \left(B_{3,n} sin(\frac{n\pi a}{L}t)\left(sin(\frac{n\pi}{L}x)\right)\right)$$

$$= \sum_{n=1}^{\infty} -\frac{2n\pi sin n\pi + 4cos n\pi - 4}{n^3\pi^3} cos(n\pi t)\left(sin(n\pi x)\right)$$

We can use the PDE solution to estimate the vibration at any point on the string. Example: The vibration results at 100×500 points of the (x,t) locations for a duration of 5s have been plotted below:

3D plot of $u_{total}(x,t)$ wrt x- & t- axes	2D plot



-Due to boundary conditions on both sides of the 1D string and the initial displacement of the rod, the vertical displacement of the string changes over time.

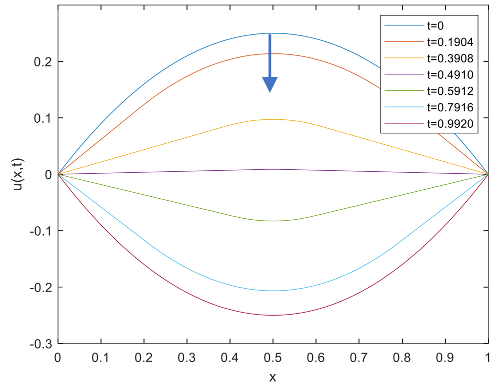
For example, $u_{total}(0.2727,1.1523) \approx -0.1751$. Note that 20 terms are used for plotting the graphs. For higher accuracy, more terms & more grid can be included but computational time will be increased.

Try to verify the answer:

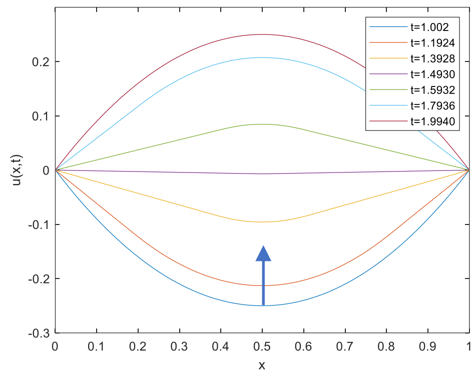
$$u_{total}(0.2727,1.1523)$$

$$\approx \sum_{n=1}^{20}\left[-\frac{2n\pi sinn\pi+4cosn\pi-4}{n^3\pi^3}\cdot cos(1.1523n\pi)\left(sin(0.2727n\pi)\right)\right]$$

From $t=0$ to $t=0.992$



From $t=1.002$ to $t=1.994$



Note that the transverse vibration solution, $u_{total}(x,t)$ due to the initial displacement does not diminish over time, this is because the original PDE equation is excluding the damping component for an ideal case with no energy loss.

Wave equation without damping component: $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$

To represent the actual system with friction/ energy loss, damping component, k can be included as such

Wave equation with damping component: $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+k\frac{\partial u}{\partial t}$

Same separation of variable method can be used to solve the damped case, thus the steps are excluded for brevity.