ENGINEERING APPLICATIONS OF ODE

WEEK 5: ENGINEERING APPLICATIONS OF ODE

5.1 MECHANICAL MASS-SPRING-DAMPER MOTIONAL SYSTEM

Example 5.1:

A mechanical system is made up three elements, the mass m which is the moving object, the spring constant k which gives the elasticity, and the damping constant c which provides damping to the system and so dampens the movement. This is usually presented schematically as in Figure 5.1.

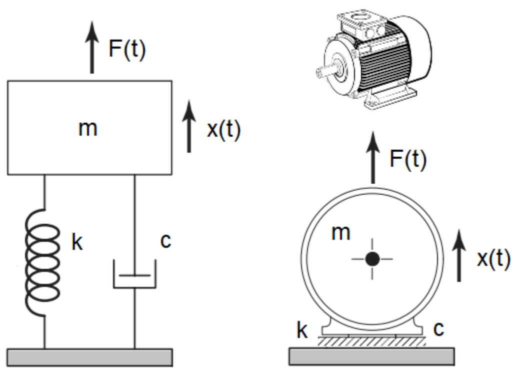


Figure 5.1. A simple mass-spring-damper system

The equation that governs the motion (displacement) $x(t)$ due to a force $F(t)$ is given as a 2nd order ODE: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$. This system is frequently used to represent vibrational or oscillatory motions. Let us consider an electric motor (Figure 5.1) with mass $m = 10$ kg. It is mounted to the ground via rubber mount with spring constant $k = 80$ N/m, damping constant $c = 40$ Ns/m. Due to uneven shaft, as the shaft rotates, the unbalanced rotation creates a cyclical force $F(t) = 20\cos 5t$ N that causes the motor to vibrate, as described by the vertical displacement $x(t)$. Find $x(t)$.

Solution:

The equation to be solved is: $10\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 80x = 20\cos 5t$

Since it is a non-homogeneous ODE, it has to be solved by:

Step 1: Homogeneous part → to get complementary solution x_c
Step 2: Non-homogeneous part → to get particular solution x_p
Step 3: Use initial condition(s) to find remaining unknowns in $x = x_c + x_p$

Step 1: Solve homogeneous part, $10\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 80x = 0$

The characteristic equation:

$10m^2 + 40m + 80 = 0$ $\qquad\qquad$ $b^2 - 4ac = 40^2 - 4(10)(80) = -1600 < 0$

We expect a pair of complex conjugate characteristic roots:

$m_{1,2} = \frac{-40 \pm \sqrt{40^2 - 4(10)(80)}}{2(10)} = -2 \pm 2i$

(Note that m refers to characteristic root here, not mass as in the question)

$x_1 = c_1 e^{(-2+2i)t}$, $\qquad x_2 = c_2 e^{(-2-2i)t}$ $\qquad \rightarrow \qquad$ $x_c = c_1 e^{(-2+2i)t} + c_2 e^{(-2-2i)t}$

However, because it is not convenient to work with complex number terms, by the Euler formula:

$e^{\pm i(2t)} = \cos(2t) + i\sin(2t)$

$x_c = e^{-2t}\left(c_1 e^{i(2t)} + c_2 e^{-i(2t)}\right)$

Eventually, the complementary solution can simply be re-stated as:

$x_c = e^{-2t}(A\cos 2t + B\sin 2t)$

Step 2: Solve non-homogeneous part, $10\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 80x = 20\cos 5t$

Method of undetermined coefficients can be used:

$RHS = 200\cos 5t$ $\qquad \rightarrow \qquad$ assume $x_p = C\cos 5t + D\sin 5t$

Differentiate: $\dot{x}_p = -5C\sin 5t + 5D\cos 5t$

$\ddot{x}_p = -25C\cos 5t - 25D\sin 5t$

The concept of solution applies. Substitute x_p and its derivatives back to the ODE for LHS = RHS:

$10(-25C\cos 5t - 25D\sin 5t) + 40(-5C\sin 5t + 5D\cos 5t) + 80(C\cos 5t + D\sin 5t) = 20\cos 5t$

$-250C\cos 5t - 250D\sin 5t - 200C\sin 5t + 200D\cos 5t + 80C\cos 5t + 80D\sin 5t = 20\cos 5t$

$(-250C + 200D + 80C)\cos 5t + (-250D - 200C + 80D)\sin 5t = 20\cos 5t \quad (+0\sin 5t)$

$(-170C + 200D)\cos 5t + (-170D - 200C)\sin 5t = 20\cos 5t \quad (+0\sin 5t)$

By comparison, we can determine the coefficients C, D:

$-170C + 200D = 20$

$-200C - 170D = 0$ $\qquad\qquad$ Solve to get: $C = -\frac{34}{689} = -0.04935$, $\qquad D = \frac{40}{689} = 0.05806$

Therefore, $x = x_c + x_p$

$x(t) = e^{-2t}(A\cos 2t + B\sin 2t) - 0.04935\cos 5t + 0.05806\sin 5t$

Note: For this question, there is no information regarding the initial conditions, such as $x(0)$ and $\dot{x}(0)$, so we will leave the solution as it is (which still contains two unknown constants, *A* & *B*, to be determined – study Example 5.2 for this kind of further steps). Because *A* & *B* have not been determined in this example (Example 5.1), the solution $x(t)$ cannot be plotted to visualize the response behavior (again, study Example 5.2 which does solve all unknowns). However, the current expression:

$$x(t) = e^{-2t}(A\cos 2t + B\sin 2t) - 0.04935\cos 5t + 0.05806\sin 5t$$

still has some usefulness. From the solution, we observe that the complementary part x_c is bounded by e^{-2t} which shows that the displacement response due to this part of the solution will diminish with time. This part (or component) of the response is called transient part. With sufficient time, the transient part will become negligible, and the subsequent displacement response will only be due to the particular part $x_p = -0.04935\cos 5t + 0.05806\sin 5t$. Therefore, for this type of analysis, sometimes the interest is only on finding x_p. Finally, it is worth mentioning that $x_p = -0.04935\cos 5t + 0.05806\sin 5t$ can be easily re-stated, using trigonometry relationship, as: $x_p = 0.07619\sin(5t - 0.7045\text{ rad})$. Since x_p is the continuous, steady-state component of the response, we will know that the amplitude of this vibration response of the electric motor is 0.07619 m (for subsequent analysis or modification).

5.2 ELECTRICAL RESISTOR-INDUCTOR-CAPACITOR (RLC) CIRCUIT

Example 5.2:

An RLC circuit is provided as shown in the Figure 5.2 where there are: an inductor of $L = 50$ Henrys (H), a resistor of $R = 5$ ohms (Ω) and a capacitor of $C = 8$ Farads (F). At $t = 0$, the switch is closed. Given the 2nd order ODE for the system is $L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = E(t)$.

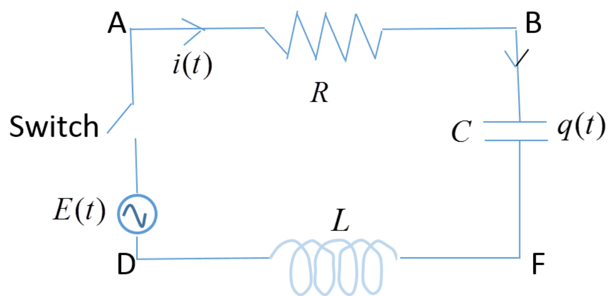


Figure 5.2. Basic RLC circuit.

Find the charge $q(t)$ and current $i(t)$ at any time $t > 0$ if the voltage is supplied by a DC battery, i.e. $E = 40$ volts (V).

Solution:

Step 1: Homogeneous Part **i.e.** $50\frac{d^2q(t)}{dt^2} + 5\frac{dq(t)}{dt} + \frac{1}{8}q(t) = 0$	**Step 2: Non-homogeneous Part** **i.e.** $50\frac{d^2q(t)}{dt^2} + 5\frac{dq(t)}{dt} + \frac{1}{8}q(t) = 40$
Characteristic equation: $$50m^2 + 5m + \frac{1}{8} = 0$$ $$400m^2 + 40m + 1 = 0$$ $$(20m + 1)^2 = 0$$ $$m_1 = m_2 = -0.05$$ ***Comment***: Repeated real root Complementary solution: $$q_c = c_1e^{-0.05t} + c_2te^{-0.05t}$$ ***Comment***: (i) $q_{c,1} = e^{-0.05t}$ and $q_{c,2} = e^{-0.05t}$ are linearly dependent. (ii) Treatment is done so that $q_{c,1} = e^{-0.05t}$ and $q_{c,2} = te^{-0.05t}$ are	The method of undetermined coefficient: $$\text{RHS} : r(x) = e^{\alpha x}P_n(x) = 40$$ $$\text{where } \alpha = 0, n = 0$$ Possible particular solution: $$q_p = e^{(0)x}Q_0(x) = A$$ Since $\alpha \neq m_1$ & m_2, treatment is not needed. ***Comment***: (i) $q_p = A$ and $q_c = c_1e^{-0.05t} + c_2te^{-0.05t}$ are linearly independent Solve the coefficient for the particular solution: $$q_p = A$$

linearly independent.	Differentiate it, we get: $$\frac{dq_p}{dt} = 0$$ $$\frac{d^2q_p}{dt^2} = 0$$ Substitute to the ODE equation: $50\frac{d^2q(t)}{dt^2} + 5\frac{dq(t)}{dt} + \frac{1}{8}q(t) = 40$ >> $50(0) + 5(0) + \frac{1}{8}A = 40$ >>$A = 320$ The actual particular solution: $$q_p = 320$$

The ***complete/ general solution*** to $50\frac{d^2q(t)}{dt^2} + 5\frac{dq(t)}{dt} + \frac{1}{8}q(t) = 40$ is

(i) ***<u>Charge solution,</u>***

$$q_{total} = q_c + q_p = c_1e^{-0.05t} + c_2te^{-0.05t} + 320$$

(ii) *Differentiate it, we obtain the **current solution***

$$i_{total} = \frac{dq_{total}}{dt} = c_2e^{-0.05t} - 0.05e^{-0.05t}(c_1 + c_2t)$$

Step 3: Solution to initial value problem

Note:

The general solution is not yet the actual solution (final solution) to the problem because it has infinite c_1, c_2 that can satisfy the problem. Recall that in the initial value problem, we can further solve the constant c_1, c_2 in the system if *the initial condition of the problem is known in advance*.

Hint for the initial condition:

'At $t = 0$, the switch is closed'

This shows that charge and current flows only when $t > 0$. Thus, the initial conditions are

$$q(0) = 0,\ i(0) = \frac{dq}{dt}_{t=0} = 0$$

Apply the initial conditions to the general solution, we obtain:

$q_{total} = q_c + q_p = c_1 e^{-0.05t} + c_2 t e^{-0.05t} + 320$

>> $q(0) = c_1 e^{-0.05(0)} + c_2(0) e^{-0.05(0)} + 320 = 0$

>> $c_1 = -320$

$i_{total} = c_2 e^{-0.05t} - 0.05 e^{-0.05t}(c_1 + c_2 t)$

>> $i_{total} = c_2 e^{-0.05(0)} - 0.05 e^{-0.05(0)}(c_1 + c_2(0)) = 0$

>> $c_2 - 0.05(c_1) = 0$

>> $c_2 = 0.05(c_1) = 0.05(-320) = -16$

The actual charge solution to the RLC circuit problem:

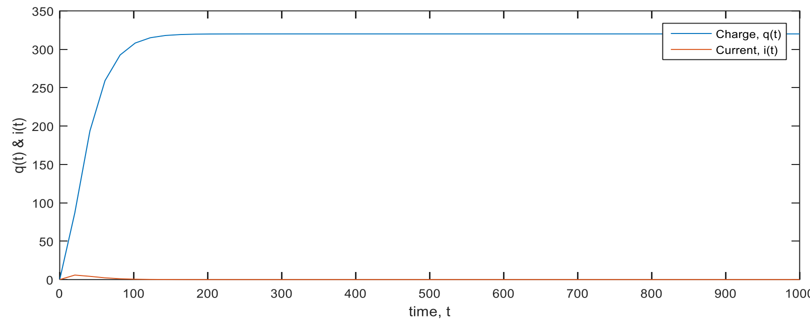
$$q_{total} = -320 e^{-0.05t} - 16 t e^{-0.05t} + 320$$

The actual current solution to the RLC circuit problem:

$$i_{total} = -16 e^{-0.05t} - 0.05 e^{-0.05t}(-320 - 16t)$$

$$i_{total} = e^{-0.05t}(16t)$$

Note: Solutions obtained by solving the problem analytically are known as analytical solutions. Once the solution expressions are known, we can plot the charge and current against time (for this RLC circuit problem with $L = 50$ H, a resistor of $R = 5\ \Omega$ and a capacitor of $C = 8$ F) as below to visualize the response behavior. The intention of this example is to *encourage student to link the mathematical result to the actual problem instead of calculating it for nothing*. However, subsequent **data analysis on specific engineering problem** requires certain knowledge on the subject and hence it is out of the scope in this study.



Hint: Relate this to the charging scenario (of battery, or capacitor in this case).

Further data analysis – Think:

(1) *Identify the transient and steady state region in the graph. Why it is important to identify them?*
Ans: *Transient region happens within around* $0 - 150s$*; while steady state region occurs after that. This relationship is important to the RLC circuit problem such as charging battery, where we can estimate how much time is needed to fully charge the battery. Moreover, it shows that the current is reduced to zero once it is fully charged.*

(2) *What are the relationships between complementary solution & particular solution with these transient and steady state region?*
Ans: *We know that the total solution is a combination of the complementary solution & particular solution. The complementary solution contributes more within the transient region and its effect diminishes within the steady state region; while the particular solution contributes significantly within the steady state region.*

(3) *Why does the charge behave as such: increase over a time initially and at time approximately 150s, the charge remains constant afterward? Why does the current behave as such: increase over a time initially and decrease after it reaches its maximum? The charge decreases to zero at time approximately 150s and remains constant afterward.*
Ans: *To understand the changes, we need to check the equation,* $q_{total} = -320e^{-0.05t} - 16te^{-0.05t} + 320$*, where complementary solution of charge,* $q_c = -320e^{-0.05t} - 16te^{-0.05t}$ *and its particular solution,* $q_p = 320$*.Since the complementary solution consists of the exponential function, i.e.* $e^{-0.05t}$*, as the time increase, this function will approach zero and thus diminish. At the same time, the particular solution remains all the time. Hence,* $q_{total} = -320e^{-0.05t} - 16te^{-0.05t} + 320$ *increase over a time initially and at time approximately 150s, the charge remains constant afterward. The current is the rate of change of the charge, i.e.* $i_{total} = \frac{dq_{total}}{dt} = e^{-0.05t}(16t)$*. Since it consists of the exponential function, it illustrates that the charging rate will be higher initially and reduce to zero afterwards, as this function will approach to zero as the time increase.*

Think: *In case you want to reduce the charging time or increase the charging capacity, what should you do?*

5.3 SOLUTION MIXING AND CONCENTRATION

Example 5.3:

A tank contains 40 liters (L) of a solution composed of 90 percent water and 10 percent alcohol. A second solution containing 50 percent water and 50 percent alcohol is added to the tank at a rate of 4 L/minute. As the second solution being added, the tank is being drained at the rate of 4 L/minute, as shown in Figure 5.3. Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after *t* minutes, how much alcohol is in the tank after 10 minutes?

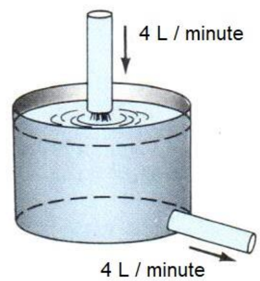


Figure 5.3. Modelling of mixture.

Solution:

Let $y(t)$ = amount of alcohol (in liters, L) in the tank at any time t (in minutes)

We can form an equation that relates y by considering the rate of change of alcohol in tank, $\frac{dy}{dt}$, which can be stated as below:

$$\frac{dy}{dt} = \text{rate of alcohol entering} - \text{rate of alcohol draining}$$

$\text{rate of alcohol entering} = 4\ L/minute \times\ 50\% = 2\ L/minute$

$\text{rate of alcohol draining} = 4\ L/minute \times \text{alcohol percentage in tank}$

But what is the alcohol percentage in tank (at any time)? This can be obtained as the amount y divided by the 40 L solution in the tank. So:

$\text{rate of alcohol draining} = 4\ L/minute \times \frac{y}{40}$

Therefore, an ODE (1st order ODE) that models this mixing scenario is established:

$$\frac{dy}{dt} = 2 - 4\left(\frac{y}{40}\right)$$

$$y' + \frac{1}{10}y = 2.$$

Let's solve the 1st order ODE by treating it as an exact differential equation: $y' + p(t)y = q(t)$. This method requires finding of integrating factor, $e^{\int p(t)dt}$

For this ODE, $p(t) = \frac{1}{10}$

Thus, the integrating factor is

$$u(t) = e^{\int p(t)dt} = e^{\int \frac{1}{10}dt} = e^{t/10}$$

Multiply the integrating factor,

$$e^{t/10}(y') + e^{t/10}\left(\frac{1}{10}y\right) = 2e^{t/10}$$

LHS is exact format,

$$\frac{d}{dt}\left[y.e^{t/10}\right] = 2e^{t/10}$$

Integrating both side with respect to *t*,

$$\int \frac{d}{dt}\left[y.e^{t/10}\right]dt = \int 2e^{t/10}dt$$

$$y.e^{t/10} = 20e^{t/10} + C$$

and the general solution is

$$y = 20 + Ce^{-t/10}$$

where *C* are arbitrary constants.

Finally, substitute initial condition to find the unknown constant *C*: *'A tank contains 40 liters (L) of a solution composed of 90 percent water and 10 percent alcohol'* Thus, initially (*t* = 0), the amount of alcohol in the tank is $y = 40L \times 10\% = 4L$, or simply $y(0) = 4$. Substitute:

$$4 = 20 + Ce^{-0/10} \qquad \Rightarrow \qquad C = -16$$

The complete solution is

$$y = 20 - 16e^{-t/10}$$

Finally, when *t* = 10 minutes, the amount of alcohol in the tank (in liters, L) is:

$$y(10) = 20 - 16e^{-10/10} = 20 - 16e^{-1} \approx 14.114$$

Note: There are multiple ways to solve this ODE: $y' + \frac{1}{10}y = 2$ is a non-homogeneous constant coefficients ODE, so it can also be solved by $y_c = Ce^{mt}$, while $y_p = A$ as from the method of undetermined coefficients. Then, $y = y_c + y_p$. Regardless of method, the final solution will be the same.

5.4 OTHER SCENARIOS

Example 5.4: Radioactive decay and radiocarbon dating

Radioactive decay is governed by the equation that belongs to 1st order ODE: $\frac{dy}{dt} = ky$, in which y represents the amount of radioactive material, k is a constant of proportionality that depends on the material. An application of this equation is in estimating the age of discovered fossils, known as radiocarbon dating, that relates to the decay of radioactive carbon-14. In living organisms, the ratio of carbon-14 to ordinary carbon-12 (representing the amount of radioactive material) is constant. When an organism dies, absorption of carbon-14 stops, so the amount (the carbon-14 to carbon-12 ratio) reduces due to radioactive decay. It is also known that the half-life of carbon-14 is 5715 years, which is the time taken for certain amount of carbon-14 to reduce to half (50 %) of the original amount.

Let us consider a scenario where a discovered fossil is measured to have carbon-14 to carbon-12 ratio of 52.5 %. With the ODE and half-life information stated above, what is the age (t, in years) of this fossil?

Solution:

We first solve the ODE to obtain a general solution of $y(t)$:

$\frac{dy}{dt} = ky$ → $\frac{dy}{dt} - ky = 0$

There are multiple ways of solving this particular 1st order ODE. For instance, we can separate the variables y & t and integrate both sides of the equation to get the solution quickly. Here, let us try the method of assuming $y(t) = Ce^{mt}$ to demonstrate this method (the same as solving 2nd order homogeneous linear ODE with constant coefficients):

Differentiate: $y(t) = Ce^{mt}$ $\quad y' = Cme^{mt}$

Substitute: $Cme^{mt} - kCe^{mt} = 0$

$$Ce^{mt}(m - k) = 0$$

Since $Ce^{mt} = y(t) \neq 0$, we obtain the characteristic equation $(m - k) = 0$, so $m = k$.

Therefore, the solution: $y(t) = Ce^{kt}$

It can be shown that the unknown constant is the initial amount of radioactive material y_0 $(t = 0)$:

Since $y = y_0$ when $t = 0$, substitute back to the solution:

$y_0 = Ce^{k0}$ → $C = y_0$

The solution: $y(t) = y_0e^{kt}$

To answer this question specifically, we first use the half-life information: $y = 0.5y_0$ when $t = 5715$:

$0.5y_0 = y_0e^{k(5715)}$ --- eqn (1) → $0.5 = e^{k(5715)}$

$k = \frac{\ln 0.5}{5715} = -0.0001213$

So: $y(t) = y_0e^{-0.0001213t}$

Since it is measured that the fossil has carbon-14 to carbon-12 ratio of 52.5 %, $y = 0.525y_0$ for this fossil at the age of t:

$0.525y_0 = y_0e^{kt} = y_0e^{-0.0001213t}$ --- eqn (2) → $0.525 = e^{-0.0001213t}$

Therefore:

$t = \frac{\ln 0.525}{-0.0001213} = 5312$ years (approximately 5300 years)

Note: For this specific question, it is not needed to know the complete solution (in particular, to determine $C = y_0$). This is because the time t for $y = 0.525y_0$ can be found with the relative information t for $y = 0.5y_0$ known – the half-life of radioactive carbon-14.In fact, if we take eqn (2) / eqn (1), the time t can be found even without finding k. However, solving the ODE to get the expression (solution) $y(t) = Ce^{kt}$ is still needed for the subsequent calculation.

Example 5.5: Newton's law of cooling

According to thermal conduction principle, the rate of change (with respect to time *t*) of the temperature *T* of an object is proportional to the difference between *T* and the temperature of the surrounding medium T_m. In mathematics, this is conveniently stated as:

$$\frac{dT}{dt} = k(T - T_m)$$

which is a 1st order ODE. Suppose that in winter the daytime indoor temperature in a certain office building is maintained at 25 deg Celsius. The heating is turned off at 10 pm (t = 0) and turned on again at 6am the following day (t = 8 hours, hr). At 2am (t = 4 hr), the indoor temperature was measured to be 20 deg Celsius. The surrounding temperature has been more or less constant at 10 deg Celsius. So, when the heating is turned on again at 6am the following day, what is the indoor temperature of the building?

Solution:

We first solve the ODE to obtain a general solution of $T(t)$:

Some information extracted:

$T_m = 10$, $\quad T(t = 0) = 25$, $\quad T(t = 4) = 20$

The 1st order ODE: $\frac{dT}{dt} = k(T - 10)$

There are several ways of solving this particular 1st order ODE. Unlike Example 5.4, this time we try the method of separating the variables to get separable differential equation instead, and integrate:

$\frac{dT}{T-10} = kdt \quad \rightarrow \quad \int \frac{dT}{T-10} = \int kdt \quad \rightarrow \quad \ln|T - 10| = kt + c$

Rearrange:

$T = e^{kt+c} + 10 \quad \rightarrow \quad T = Ae^{kt} + 10$ Note: $A = e^c = constant$

Substitute $T(t = 0) = 25$:

$25 = Ae^{k(0)} + 10 \quad \rightarrow \quad A = 15$

Substitute $T(t = 4) = 20$:

$20 = 15e^{k(4)} + 10 \quad \rightarrow \quad k = \left(\frac{1}{4}\right) \ln\left(\frac{20-10}{15}\right) = -0.1014$

So, the complete solution: $T = 15e^{-0.1014t} + 10$

Once the complete solution is found, the answer to the question is straight-forward:

At 6am the following day (t = 8 hr),

$T = 15e^{-0.1014(8)} + 10 = 16.665$ deg Celsius

Summary note: from the various examples, it can be seen that quite a number of engineering or physical scenarios are modelled by ODE, whether 1st order or 2nd order. Therefore, solving differential equation is very important in understanding the behavior of the response of a system that corresponds to an input / excitation.

The mathematic modelling (deriving the equation) of a system involves understanding of specific law and theory, thus it is not within the scope of this study. However, students are expected to know how to solve the given 1st order & 2nd order ODE for various engineering applications.

Details of some mathematic modelling and some further examples are available in Appendices 5.1 – 5.7. This includes:

(i) Mathematic modelling of a liquid system (*1st order ODE*) (Appendix 5.1)
(ii) Mathematic modelling of an RLC electrical circuit (*2nd order ODE*) (Appendix 5.2)
(iii) Mathematic modelling of an RLC electrical circuit under electromotive force excitation (*2nd order ODE*) (Appendix 5.3)
(iv) Mathematic modelling of a vibrating spring without damping (*2nd order ODE*) (Appendix 5.4)
(v) Mathematic modelling of a vibrating spring with damping (*2nd order ODE*) (Appendix 5.5)
(vi) Mathematic modelling of a damped vibration system under forced vibration (*2nd order ODE*) (Appendix 5.6)
(vii) Analogue between vibrational system and electrical circuit (Appendix 5.7)