LAPLACE TRANSFORM

WEEK 8: LAPLACE TRANSFORM

8.1 INTRODUCTION

Laplace transforms are invaluable for any engineer's mathematical toolbox as they make solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. The key motivation for learning about Laplace transforms is that the process of solving an ODE is simplified to an algebraic problem (and transformations). Applications abound: electrical networks, springs, mixing problems, signal processing, and other areas of engineering and physics.

8.2 DEFINITION

Laplace transform of a function *f(t)* is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

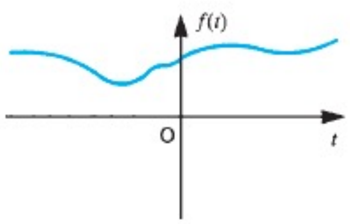
Laplace transform is called an **integral transform** because it transforms (changes) a function in one space to a function in another space by a *process of integration* that involves a kernel, $k(s,t) = e^{-st}$.

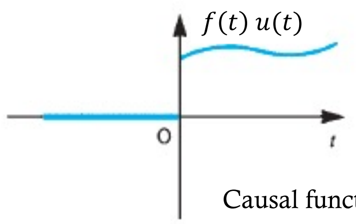
$$F(s) = \int_0^\infty k(s,t)f(t)\ dt$$

One can imagine $f(t)$ to be a function in the time domain (t). By performing Laplace transform, one can transform the time domain function to the frequency doman $F(s)$ to be in frequency domain (s).

The purpose of doing that is because it is easier to solve integrals and ordinary differential equations with constant coefficients in the frequency domain than it is in the time domain. Once solved (easily) in the frequency domain, one needs to revert it back to the time domain (using inverse Laplace transform) to obtain the solution. This will be topic to cover in Week 9, but before doing so, one needs to be familiar with the mechanics of forward and inverse Laplace transform.

Transforming f(t) to a causal function:





Causal function

Note that since the limit of the above integral begins at t = 0, the behavior of the function $f(t)$ at t < 0 is ignored. This is so-called causal function (where $f(t) = 0$ at any value $t < 0$). In other words, we only deal with real time, $t \geq 0$.

Example 8.1:

1. Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$.

Solution:

$$F(s) = \mathcal{L}(1)$$

$$= \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^\infty$$

$$= \frac{1}{s}$$

2. Let $f(t) = e^{at}$ when $t \geq 0$, where a is a constant. Find $\mathcal{L}\{f(t)\}$.

Solution:

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} \cdot e^{-st}\, dt$$

$$= -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$

8.3 LINEARITY

For any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b, the transform of $af(t) + bg(t)$ is given by

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Proof:

$$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}\big(af(t) + bg(t)\big)\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$$

$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Example 8.2:

1. Find the Lapalce transforms of $\cosh(at)$ and $\sinh(at)$.

Solution:

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2-a^2}$$

$$\mathcal{L}\{\sinh(at)\} = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{a}{s^2 - a^2}$$

8.3.1 LAPLACE TRANSFORM PAIRS

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Exercise

Based on the Laplace transform pairs, compute the Laplace transform of the following functions, $f(t)$:

(i) 4 Ans: $\frac{4}{s}$

(ii) 10 t^6 Ans: $\frac{7200}{s^7}$

(iii) $20 \sin 5t$ Ans: $\frac{100}{s^2+25}$

(iv) $7 \cos(\frac{t}{2})$ Ans: $\frac{28s}{4s^2+1}$

(v) $2 \sin^2 6t$ Ans: $\frac{1}{s} - \frac{s}{s^2+144}$

8.4 INVERSE TRANSFORM

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathcal{L}\{f(t)\} = F(s)$, we say $f(t)$ is the inverse Laplace transform of $F(s)$.

The inverse Laplace transform is denoted as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Note: $\mathcal{L}^{-1}\{\mathcal{L}(f(t))\} = f(t)$ and $\mathcal{L}\{\mathcal{L}^{-1}(F(s))\} = F(s)$

In determining the inverse Laplace transform, some manipulations must be done to get $F(s)$ into a form suitable for the direct use of the Laplace transform table.

Example 8.3:

Evaluate (a) $\mathcal{L}^{-1}\left(\frac{1}{s^5}\right)$ (b) $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right)$ (c) $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right)$.

Solution:

(a) $\mathcal{L}^{-1}\left(\frac{1}{s^5}\right) = \frac{1}{4!}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) = \frac{1}{24}t^4$

(b) $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right) = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{1}{\sqrt{7}}\sin\sqrt{7}t$

(c) $\mathcal{L}^{-1}\left(\frac{-2s+6}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right) = -2\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{6}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = -2\cos 2t + 3\sin 2t$

8.5 PARTIAL FRACTION

The solution $F(s)$ usually comes out as a general form of

$$F(s) = \frac{P(s)}{Q(s)} = S(s) + \frac{R(s)}{Q(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}$$

where $P(s)$ and $R(s)$ are the numerators and $Q(s)$ is the denominator.

A proper rational function of $F(s)$ should be expanded as a sum of partial fraction before its inverse Laplace transform can be found.

Depending on the roots of $Q(s)$ we have fraction expansion in the form:

Denominator $Q(s)$	Example of $F(s)$	Partial fraction expansion
1. Distinct & real roots $Q(s) = (a_1 s + b_1)(a_2 s + b_2) \dots (a_k s + b_k)$	$\frac{96s}{s(s+8)(s+6)}$	$\frac{A}{s} + \frac{B}{s+8} + \frac{C}{s+6}$
2. Repeated & real roots $Q(s) = (a_1 s + b_1)^r$	$\frac{s+30}{(s+3)^2}$	$\frac{A}{(s+3)^2} + \frac{B}{s+3}$
3. Distinct irreducible quadratic factors $Q(s) = as^2 + bs + c$, where $b^2 - 4ac < 0$	$\frac{s+3}{s^2+6s+25}$	$\frac{As+B}{s^2+6s+25}$
4. Repeated irreducible quadratic factors $Q(s) = (as^2 + bs + c)^r$, where $b^2 - 4ac < 0$	$\frac{s+6}{(s^2+6s+25)^2}$	$\frac{As+B}{s^2+6s+25} + \frac{Cs+D}{(s^2+6s+25)^2}$

where A, B, C and D are constants.

Example 8.4:

Find the inverse Laplace transform of $F(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$.

Solution:

$$F(s) = \frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{s+2}$$

$$10s^2 + 4 = A(s+1)(s+2)^2 + Bs(s+2)^2 + Cs(s+1) + Ds(s+1)(s+2)$$

If we set $s = 0$, $s = -1$, $s = -2$ and $s = 1$, we obtain

$$A = 1,\ B = -14,\ C = 22,\ D = 13 \text{ respectively.}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{10s^2+4}{s(s+1)(s+2)^2}\right\} = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{14}{s+1} + \frac{22}{(s+2)^2} + \frac{13}{s+2}\right) = 1 - 14e^{-t} + 22te^{-2t} + 13e^{-2t}$$

Exercise

Solve the inverse Laplace transform of the followings:

(i) $\mathcal{L}^{-1}\left\{\frac{4s^2-5s-41}{(s-1)(s+2)(s-3)}\right\}$ Ans: $7e^t - e^{-2t} - 2e^{3t}$

(ii) $\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\}$ Ans: $1 - \cos t$

(iii) $\mathcal{L}^{-1}\left\{\frac{s^2+s-52}{(s+1)(s^2+25)}\right\}$ Ans: $-2e^{-t} + 3\cos 5t - \frac{2}{5}\sin 5t$

8.6 LAPLACE TRANSFORM OF DERIVATIVES

8.6.1 LAPLACE TRANSFORM OF FIRST DERIVATIVE

If $f(t)$ is continuous for all $t \geq 0$, satisfies the growth condition and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= e^{-st} f(t)\big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

8.6.2 LAPLACE TRANSFORM OF SECOND DERIVATIVE

If $f(t)$ and $f'(t)$ are continuous for all $t \geq 0$, satisfies the growth condition and $f''(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Proof:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$
$$= s\big(s\mathcal{L}\{f(t)\} - f(0)\big) - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

8.6.3 LAPLACE TRANSFORM OF THE DERIVATIVE $f^{(n)}$ OF ANY ORDER

If $f, f', \dots f^{(n-1)}$ are continuous for all $t \geq 0$, satisfies the growth condition and $f^{(n)}$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

8.7 LAPLACE TRANSFORM OF INTEGRAL

If $f(t)$ is piecewise continuous for all $t \geq 0$,

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

Proof:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}' = s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} - \int_0^0 f(\tau)\,d\tau$$

$$\mathcal{L}\{f(\tau)\} = s\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(\tau)\}$$

Example 8.5:

Find the inverse of $\frac{1}{s(s^2+\omega^2)}$ and $\frac{1}{s^2(s^2+\omega^2)}$.

Solution:

We know that $\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin\omega t}{\omega} = f(\tau)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\big(f(\tau)\big)\right\}$$

$$= \int_0^t \frac{\sin \omega\tau}{\omega} d\tau$$

$$= \frac{1}{\omega^2}(1 - \cos \omega t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega\tau)\, d\tau$$

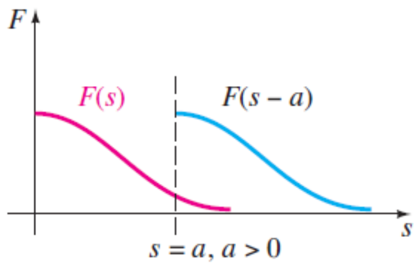
$$= \frac{t}{\omega^2} - \frac{\sin \omega t}{\omega^3}$$

8.8 FIRST SHIFT THEOREM: s-SHIFTING

If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

where $(s - a) > k$.



Proof:

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= \int_0^\infty \left(e^{at}f(t)\right)e^{-st}\, dt$$

$$= \mathcal{L}\{e^{at}f(t)\}$$

Example 8.6:

1. From previous example, we know that $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$ then

   $$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$

   $$\mathcal{L}\{e^{at}\sin(\omega t)\} = \frac{\omega}{(s-a)^2+\omega^2}$$

2. Using first shift theorem, find the inverse of the transform $\mathcal{L}\{f(t)\} = \frac{3s-137}{s^2+2s+401}$.

Solution:

$$\mathcal{L}^{-1}\left(\frac{3s-137}{s^2+2s+401}\right) = \mathcal{L}^{-1}\left(\frac{3(s+1)-140}{(s+1)^2+400}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+400}\right) - 7\mathcal{L}^{-1}\left(\frac{20}{(s+1)^2+400}\right)$$

$$= 3e^{-t}\cos 20t - 7\,\mathrm{e}^{-t}\sin 20t$$

Exercise

Solve the inverse Laplace transform of the followings:

(i) $\mathcal{L}^{-1}\left\{\frac{6}{s^2+8s+25}\right\}$ Ans: $2\,e^{-4t}\sin 3t$

(ii) $\mathcal{L}^{-1}\left\{\frac{3s-5}{s^2+2s+5}\right\}$ Ans: $3e^{t}\cos 2t - 4\,\mathrm{e}^{-\mathrm{t}}\sin 2t$

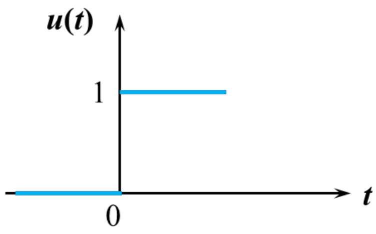
(iii) $\mathcal{L}^{-1}\left\{\frac{s}{(s+5)^4}\right\}$ Ans: $\frac{1}{2}e^{-5t}t^2 - \frac{5}{3!}e^{-5t}t^3$

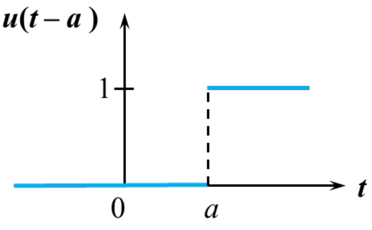
8.9 UNIT STEP FUNCTION (HEAVISIDE FUNCTION)

Mathematical definition of a unit step function is

$$u(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>a\end{cases}$$

The unit step function has a discontinuity, or jump, at the origin for $u(t)$ or at the position a for $u(t-a)$ where a is an arbitrary positive.





The transform of $u(t-a)$ follows directly from the defining integral:

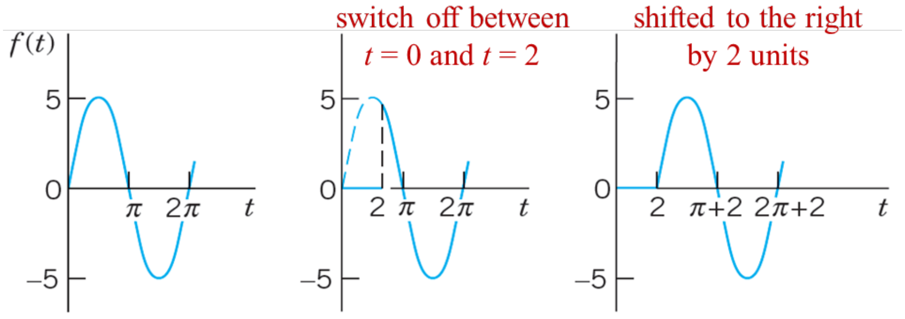
$$\mathcal{L}\{u(t-a)\} = \int_0^{\infty} e^{-st}\,u(t-a)dt$$

$$= \int_a^{\infty} e^{-st}\cdot 1\,dt$$

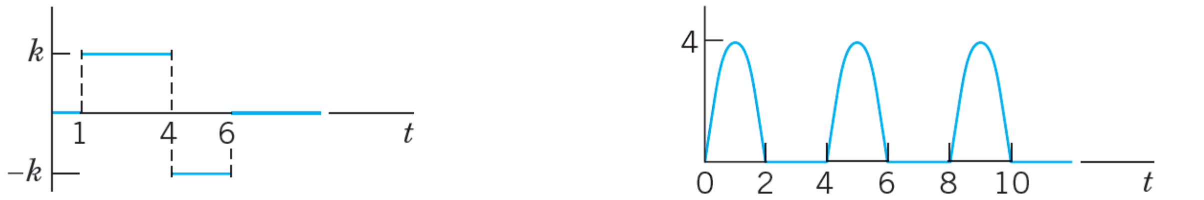
$$= -\frac{1}{s}e^{-st}\Big|_a^{\infty}$$

$$= \frac{e^{-as}}{s}$$

Example 8.7:



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$



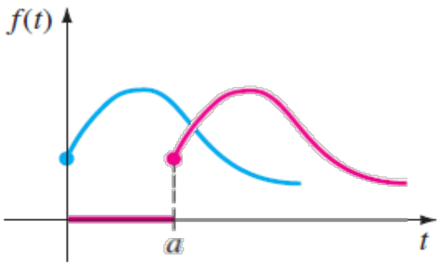
(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

8.10 SECOND SHIFT THEOREM: TIME SHIFTING (t-SHIFTING)

If $f(t)$ has the transform $F(s)$, then the "shifted-function"

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$$



Proof:

Let $\tau = t - a$,

$$\mathcal{L}\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

$$= \int_{-a}^\infty e^{-s(\tau+a)} f(\tau)u(\tau)d\tau$$

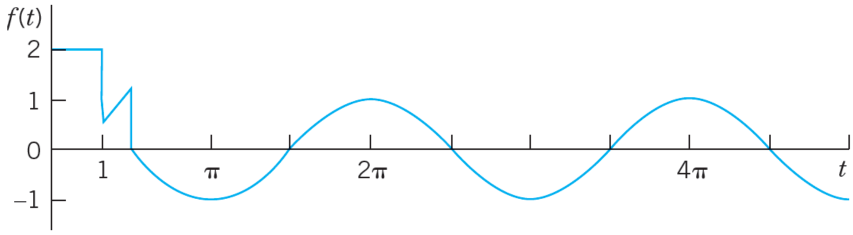
$$= e^{-as}\int_0^{\infty} e^{-s\tau} f(\tau)d\tau$$

$$= e^{-as}F(s)$$

Example 8.8:

1. Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } t > \frac{1}{2}\pi \end{cases}$$



Solution:

$$f(t) = \underbrace{2\big(u(t) - u(t-1)\big)}_{part\ (a)} + \frac{1}{2}t^2\left(\underbrace{u(t-1)}_{part\ (b)} - \underbrace{u\left(t - \frac{1}{2}\pi\right)}_{part\ (c)}\right) + \underbrace{(\cos t)\, u\left(t - \frac{1}{2}\pi\right)}_{part\ (d)}$$

Part (a): $\mathcal{L}\{2\big(u(t) - u(t-1)\big)\} = 2\left(\frac{1}{s} - \frac{e^{-s}}{s}\right)$

Part (b): $\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \frac{1}{2}\mathcal{L}\{(t-1)^2u(t-1) + 2(t-1)\,u(t-1) + u(t-1)\}$

$$= \frac{1}{2}e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

$$= e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

Part (c):

$$\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} = \frac{1}{2}\mathcal{L}\left\{\left(t - \frac{1}{2}\pi\right)^2 u\left(t - \frac{1}{2}\pi\right) + \pi\left(t - \frac{1}{2}\pi\right) u\left(t - \frac{1}{2}\pi\right) + \frac{1}{4}\pi^2\, u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$= \frac{1}{2}e^{-\pi s/2}\left(\frac{2}{s^3} + \frac{\pi}{s^2} + \frac{\pi^2}{4s}\right)$$

$$= e^{-\pi s/2}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)$$

Part (d): $\mathcal{L}\left\{(\cos t)\, u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\sin\left(t - \frac{1}{2}\pi\right)\, u\left(t - \frac{1}{2}\pi\right)\right\}$

$$= -\frac{e^{-\pi s/2}}{s^2+1}$$

Combining all the terms:

$$\mathcal{L}\{f(t)\} = \frac{2}{s} - \frac{2}{s}e^{-s} + e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right) - e^{-\frac{\pi s}{2}}\left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right) - \frac{e^{-\pi s/2}}{s^2+1}$$

2. Find the inverse transform $f(t)$ of $F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$.

Solution:

First, we consider without the exponential functions in the numerator.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{\sin \pi t}{\pi}, \qquad \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = te^{-2t}$$

By second shift theorem,

$$f(t) = \frac{1}{\pi}\sin\big(\pi(t-1)\big)\,u(t-1) + \frac{1}{\pi}\sin\big(\pi(t-2)\big)\,u(t-2) + (t-3)e^{-2(t-3)}u(t-3)$$