LAPLACE TRANSFORM SOLUTIONS FOR DIFFERENTIAL EQUATIONS

WEEK 9: LAPLACE TRANSFORM SOLUTIONS FOR DIFFERENTIAL EQUATIONS

9.1 SOLVING LINEAR ODES

Having mastered the mechanics of forward and inverse Laplace transform, this Chapter applies such skills to solve linear differential equations.

The Laplace transform of a linear differential equation (in t-domain) with constant coefficient yields an algebraic equation, Y(s) in s-domain, which can be solved easily. Once solved, the solution is s-domain can be easily reconverted to t-domain to obtain the final solution.

Initial condition(s) is(are) required to solve differential equations using Laplace transform.

Example 9.1: Solving first-order initial value problem

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13\sin 2t\,,\ \ y(0) = 6$$

Solution:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\} \qquad (1)$$

From Table of Laplace Transform,

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 6 \quad \text{and} \quad \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

Equation (1) becomes

$$sY(s) - 6 + 3Y(s) = 13\left(\frac{2}{s^2+4}\right)$$

$$(s+3)Y(s) = 6 + 13\left(\frac{2}{s^2+4}\right)$$

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

Performing partial fraction:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$A = 8, B = -2, C = 6$$

$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$

Therefore, by inverse Laplace transform,

$y(t) = \mathcal{L}^{-1}\left\{\frac{8}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$

Example 9.2: Solving second-order initial value problem

Solve $y" - 3y' + 2y = e^{-4t},\ y(0) = 1,\ y'(0) = 5.$

Solution:

$\mathcal{L}\{y"\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$

$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$

$s^2Y(s) - s - 5 - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s+4}$

$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$

$Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$

Performing partial fraction:

$Y(s) = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16}{5}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$

Therefore, by inverse Laplace transform,

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

Exercise

Solve the following linear differential equations using Laplace transform method

(i) $y'' + 14\,y' + 49\,y = 40\,t^3e^{-7t}$ where, $y'(0) = -5$ and $y(0) = 2$

Ans: $y(t) = e^{-7t}(2t^5 + 9t + 2)$

(ii) $y'' + y' - 2y = 4t$ where, $y'(0) = 0$ and $y(0) = 1$

Ans: $y(t) = -1 - 2t + 2e^{t}$

9.2 DIFFERENTIATION OF TRANSFORMS

If $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$, then

$$F'(s) = -\int_0^\infty tf(t)e^{-st}\,dt = \mathcal{L}\{-tf(t)\}$$

Proof:

$F(s) = \int_0^\infty f(t)e^{-st}\,dt$

$F'(s) = \frac{d}{ds}\left(\int_0^\infty f(t)e^{-st}\,dt\right)$

$= \int_0^\infty \frac{d}{ds}(e^{-st})\,f(t)\,dt$

$= \int_0^\infty -te^{-st}\,f(t)\,dt$

$= \mathcal{L}\{-tf(t)\}$

In other words, if $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

Example 9.3:

1. Given $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2} = F(s)$,

   Then $\quad \mathcal{L}\{-t\sin(\omega t)\} = F'(s) = -\frac{2s\omega}{(s^2+\omega^2)^2}$

2. Find the inverse transform of $\ln\left(1+\frac{\omega^2}{s^2}\right)$.

Solution:

Let $F(s) = \ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\left(\frac{s^2+\omega^2}{s^2}\right) = \ln(s^2+\omega^2) - \ln(s^2)$

$F'(s) = \frac{d}{ds}(\ln(s^2+\omega^2) - \ln(s^2)) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2} = \mathcal{L}\{-tf(t)\}$

Taking inverse transform,

$\mathcal{L}^{-1}\{F'(s)\} = -tf(t) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}\right)$

$-tf(t) = 2\cos(\omega t) - 2$

$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = f(t) = \frac{2}{t}(1-\cos(\omega t))$

9.3 INTEGRATION OF TRANSFORMS

If $f(t)$ satisfies the assumptions of the existence theorem and the limit of $f(t)/t$ exists when t approaches 0 from the right, then

$$\mathcal{L}^{-1}\left(\int_s^\infty F(\tilde{s})\,d\tilde{s}\right) = \frac{f(t)}{t}$$

Proof:

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty\left(\int_0^\infty e^{-\tilde{s}t} f(t)\,dt\right)d\tilde{s}$$

$$= \int_0^\infty\left(\int_s^\infty e^{-\tilde{s}t}\,d\tilde{s}\right)f(t)\,dt$$

$$= \int_0^\infty\left(-\frac{1}{t}e^{-\tilde{s}t}\Big|_s^\infty\right)f(t)\,dt$$

$$= \int_0^\infty \frac{1}{t}e^{-st} f(t)\,dt$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

9.4 DIRAC DELTA FUNCTION

Mechanical systems are often acted on by an external force (or electromotive force in an electrical circuit) of large magnitude that acts only for a very short period of time. We can model such phenomena and problems by "Dirac delta function," and solve them very effectively by the Laplace transform.

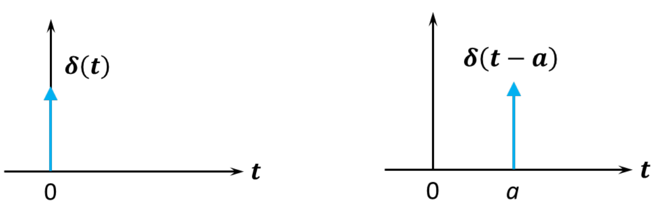
To model situations of that type, we consider the function

$$f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$

As $k \to 0$, this limit is denoted by $\delta(t-a)$, that is

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$

$\delta(t-a)$ is called the Dirac delta function or the unit impulse function.



The Laplace transform of the Dirac delta function is given by

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \qquad \text{for } a > 0$$

Example 9.4:

Solve $y'' + y = 4\delta(t - 2\pi)$ subject to $y(0) = 1, y'(0) = 0$.

Solution:

The Laplace transform of the differential equation is

$s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$

$(s^2 + 1)Y(s) - s = 4e^{-2\pi s}$

$Y(s) = \frac{4e^{-2\pi s}+s}{s^2+1}$

$y(t) = 4\sin(t - 2\pi)\,u(t - 2\pi) + \cos t$

$$y(t) = \begin{cases} \cos t & 0 \le t \le 2\pi \\ 4\sin t + \cos t & t \ge 2\pi \end{cases}$$

Exercise

Solve the linear differential equation using Laplace transform method: $y' + y = \delta(t - 1)$, where $y(0) = 1$

Ans: $y(t) = e^{-(t-1)}u(t - 1) + e^{-t}$

9.5 CONVOLUTION

The convolution of two functions *f*(*t*) and *g*(*t*) is denoted by the standard notation $f * g$ and defined by the integral $(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$.

The Laplace transform is given by $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$.

Proof: $F(s)G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-s\sigma} g(\sigma)\,d\sigma\right)$

$= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+\sigma)} g(\sigma)\,d\sigma\right) f(\tau)\,d\tau$

$= \int_0^\infty \left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right) f(\tau)\,d\tau \qquad [t = \sigma + \tau]$

$= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right) dt$

$= \mathcal{L}\{(f * g)\}$

9.5.1 PROPERTIES OF CONVOLUTION

i. Commutative law: $f * g = g * f$

ii. Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$

iii. Associative law: $(f * g) * v = f * (g * v)$

iv. $f * 0 = 0 * f = 0$

v. $f * 1 \neq f$

9.5.2 INTEGRAL EQUATIONS

Convolution also helps in solving certain integral equations, that is, equations in which the unknown function *y*(*t*) appears in an integral.

Example 9.5: Volterra integral equation of the second kind

Solve $y(t) - \int_0^t y(\tau) \sin(t - \tau)\, d\tau = t$.

Solution:

$y - y * \sin t = t$

Applying Laplace transform and convolution theorem, we obtain

$Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2}$

$Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}$

$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$

Therefore, $y(t) = t + \frac{t^3}{6}$

9.6 SYSTEM OF ODEs

We consider a first-order linear system with constant coefficients:

$$y_1' = a_{11}y_1 + a_{12}y_2 + g_1$$

$$y_2' = a_{21}y_1 + a_{22}y_2 + g_2$$

If we transform it,

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1$$

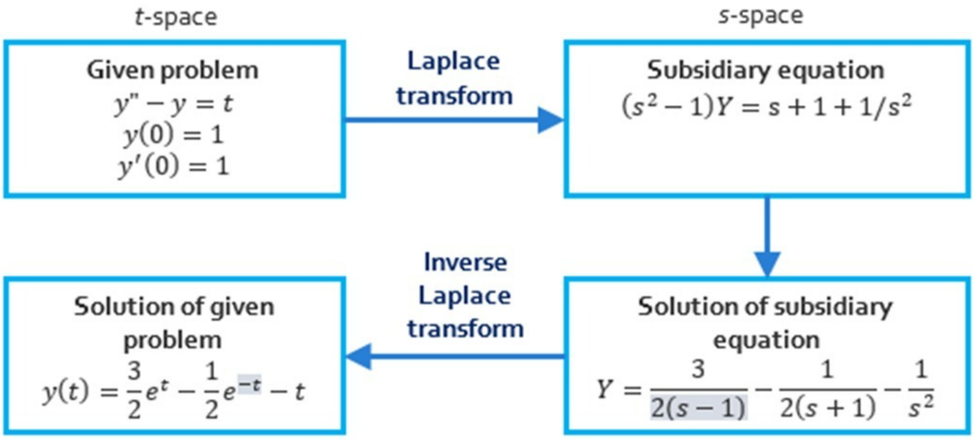
$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2$$

By collecting the Y_1- and Y_2-terms we have

$$(a_{11} - s)Y_1 + a_{12}Y_2 = -y_1(0) - G_1(s)$$

$$a_{21}Y_1 + (a_{22} - s)Y_2 = -y_2(0) - G_2(s)$$

By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution y_1 and y_2 of the system.

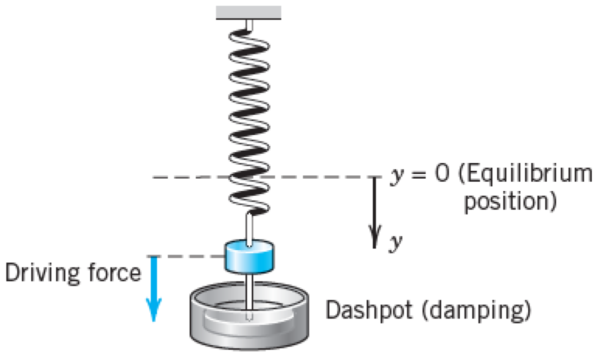


Steps of the Laplace transform method

Example 9.6:

1. Damped Forced Vibrations

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval.



Mechanical system

$y'' + 2y' + 2y = r(t),\ \ r(t) = 10 \sin 2t$ if $0 < t < \pi$ and 0 if $t > \pi$;

$y(0) = 1,\ y'(0) = -5$

Solution:

$$y'' + 2y' + 2y = 10\sin 2t\big(u(t) - u(t-\pi)\big)$$

Using Laplace transform,

$$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\frac{2}{s^2+4}(1 - e^{-\pi s})$$

$$(s^2 + 2s + 2)Y = s - 3 + 10\frac{2}{s^2+4}(1 - e^{-\pi s})$$

$$Y = \underbrace{\frac{s-3}{(s^2+2s+2)}}_{part\ (a)} + \underbrace{\frac{20}{(s^2+2s+2)(s^2+4)}}_{part\ (b)} - \underbrace{\frac{20e^{-\pi s}}{(s^2+2s+2)(s^2+4)}}_{part\ (b1)}$$

Applying inverse Laplace transform,

Part (a): $\mathcal{L}^{-1}\left(\frac{s-3}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{(s+1)-4}{(s+1)^2+1}\right)$

$$= e^{-t}(\cos t - 4\sin t)$$

Part (b): Partial fraction expansion: $\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{As+B}{(s+1)^2+1} + \frac{Ms+N}{s^2+4}$

$$20 = (As + B)(s^2 + 4) + (Ms + N)(s^2 + 2s + 2)$$

$$20 = (A + M)s^3 + (2M + B + N)s^2 + (4A + 2M + 2N)s + (4B + 2N)$$

Equating the coefficients of each power of *s* on both sides gives the four equations:

$A + M = 0;$ $\qquad$ $2M + B + N = 0;$

$4A + 2M + 2N = 0;$ $\qquad$ $4B + 2N = 20;$

We determine *A* = 2, *B* = 6, *M* = -2, *N* = -2

$$\frac{20}{(s^2+2s+2)(s^2+4)} = \frac{2s+6}{(s+1)^2+1} - \frac{(2s+2)}{s^2+4} = \frac{2(s+1)+4}{(s+1)^2+1} - \frac{2s+2}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{20}{(s^2+2s+2)(s^2+4)}\right\} = e^{-t}(2\cos t + 4\sin t) - 2\cos 2t - \sin 2t$$

Part (b1): From second shift theorem, we have

$$\mathcal{L}^{-1}\left\{\frac{20e^{-\pi s}}{(s^2+2s+2)(s^2+4)}\right\} = e^{-(t-\pi)}(2\cos(t-\pi) + 4\sin(t-\pi)) - 2\cos 2(t-\pi) - \sin 2(t-\pi)$$

$$= e^{-(t-\pi)}(-2\cos t - 4\sin t) - 2\cos 2t - \sin 2t$$

$\cos(t-\pi) = -\cos t$
$\sin(t-\pi) = -\sin t$

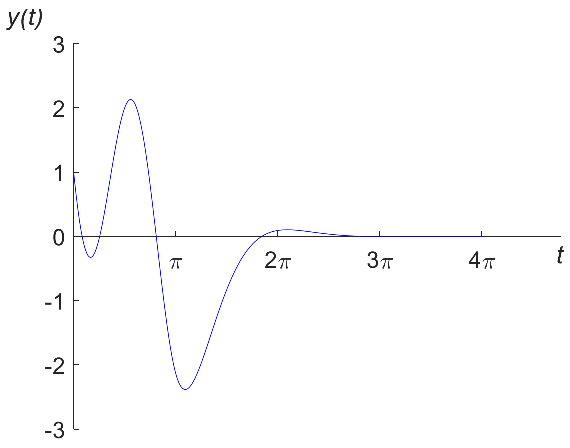
Therefore, the solution is

$$y(t) = e^{-t}(\cos t - 4\sin t) + e^{-t}(2\cos t + 4\sin t) - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi$$

$$= 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi$$

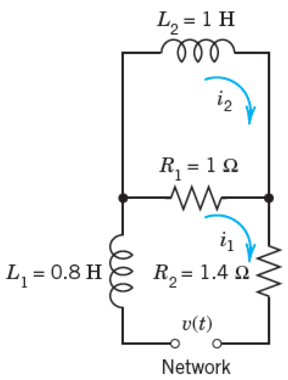
$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t - \left[e^{-(t-\pi)}(-2\cos t - 4\sin t) - 2\cos 2t - \sin 2t\right] \qquad \text{if } t > \pi$$

$$= e^{-t}\left((3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t\right) \qquad \text{if } t > \pi$$



2. **Electrical Network**

Find the currents $i_1(t)$ and $i_2(t)$ in the network with L and R measured in terms of the usual units, $v(t) = 100$ volts if $0 \le t \le 0.5$ sec and 0 thereafter, and $i(0) = 0$, $i'(0) = 0$.



Solution:

The model of the network is obtained from Kirchhoff's Voltage Law:

For the lower circuit:

$$0.8i_1' + 1(i_1 - i_2) + 1.4i_1 - 100[1 - u(t - 0.5)] = 0$$

For the upper circuit:

$$1i_2' + 1(i_2 - i_1) = 0$$

Applying Laplace transform,

$$0.8sI_1 + (I_1 - I_2) + 1.4I_1 = 100\left[\frac{1}{s} - \frac{e^{-0.5s}}{s}\right]$$

$$sI_2 + (I_2 - I_1) = 0$$

Solving algebraically for I_1 and I_2:

$$I_1 = \left(\frac{500}{7s} - \frac{125}{3(s+0.5)} - \frac{625}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

$$I_2 = \left(\frac{500}{7s} - \frac{250}{3(s+0.5)} + \frac{250}{21(s+3.5)}\right)(1 - e^{-0.5s})$$

The inverse transform for $0 \le t \le 0.5$

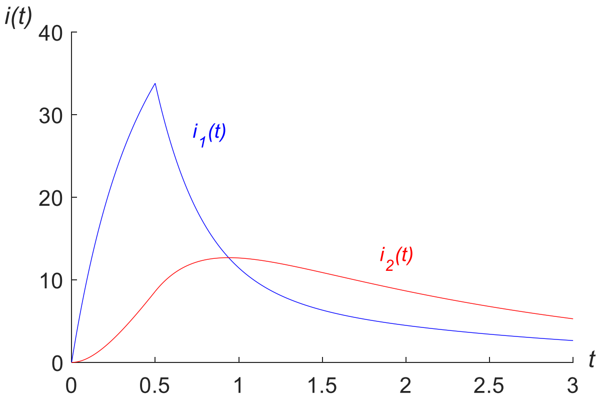
$$i_1(t) = \frac{500}{7} - \frac{125}{3}e^{-0.5t} - \frac{625}{21}e^{-3.5t}$$

$$i_2(t) = \frac{500}{7} - \frac{250}{3}e^{-0.5t} + \frac{250}{21}e^{-3.5t}$$

The inverse transform for $t > 0.5$

$$i_1(t) = i_1(t) - i_1(t - 0.5) = -\frac{125}{3}(1 - e^{0.25})e^{-0.5t} - \frac{625}{21}(1 - e^{1.75})e^{-3.5t}$$

$$i_2(t) = i_2(t) - i_2(t - 0.5) = -\frac{250}{3}(1 - e^{0.25})e^{-0.5t} + \frac{250}{21}(1 - e^{1.75})e^{-3.5t}$$



Exercise

Solve the following ODEs using Laplace transform method:

$y_1' = 5y_1 + y_2$ and $y_2' = y_1 + 5y_2 + 2\,\delta(t-1)$

Initial conditions: $y_1(0) = 0;\ y_2(0) = 10$

Ans:

$y_1 = 5e^{6t} - 5e^{4t} + \{e^{6(t-1)} - e^{4(t-1)}\}u(t-1)$ and $y_2 = 5e^{6t} + 5e^{4t} + \{e^{6(t-1)} - e^{4(t-1)}\}u(t-1)$

Table of Laplace Transform

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$af(t)+bg(t)$	$aF(s)+bG(s)$
$\delta(t)$	1	$u(t-a)$	$\frac{e^{-as}}{s}$
t	$\frac{1}{s^2}$	$\delta(t-a)$	e^{-as}
$t^n,\ \ n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	$f(t-a)u(t-a)$	$e^{-as}F(s)$
e^{at}	$\frac{1}{s-a}$	$e^{at}f(t)$	$F(s-a)$
te^{at}	$\frac{1}{(s-a)^2}$	$\frac{df}{dt}$	$sF(s)-f(0)$
$t^n e^{at},\ \ n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	$\frac{d^2f}{dt^2}$	$s^2F(s)-sf(0)-f'(0)$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{d^nf}{dt^n}$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}F(s)$
$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$	$tf(t)$	$-\frac{d}{ds}F(s)$
$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$	$f(t)*g(t)$	$F(s)G(s)$
$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$		